

Set Programming: Theory and Computation

Private Defense

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Guarantees

Geometric duality

Applications

$$\begin{array}{ll} \min_{S_1 \subseteq \mathbb{R}^{n_1}, \dots, S_N \subseteq \mathbb{R}^{n_N}} & f(S_1, \dots, S_N) \\ \text{subject to:} & g_i(S_1, \dots, S_N) \subseteq h_i(S_1, \dots, S_N), \quad i = 1, 2, \dots, M. \end{array}$$

Linear inclusion

$$\begin{aligned} AS_i &\subseteq S_j, \\ S_i &\subseteq A^{-1}S_j. \end{aligned}$$

Algebraic inclusion

$$\begin{aligned} S_i &\subseteq \pi_{[n_i], n_i+m} \begin{bmatrix} A & B \end{bmatrix}^{-1} S_j, \\ CS_i &\subseteq ES_j. \end{aligned}$$

Guarantees

Löwner-John ellipsoid of intersection of ellipsoids

John ellipsoid:

$$\begin{array}{ll} \max_{\mathcal{E}_P \subseteq \mathbb{R}^n} & \text{vol}(\mathcal{E}_P) \\ \text{subject to:} & \mathcal{E}_P \subseteq \mathcal{C}. \end{array}$$

Theorem: Unique $\mathcal{E}_P^*$ and ($\mathcal{C}$ symmetric):

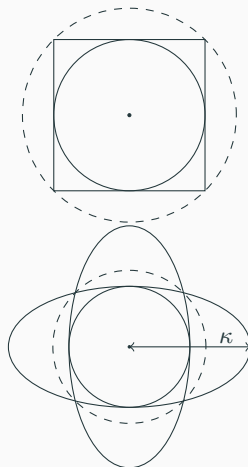
$$\mathcal{E}_P^* \subseteq \mathcal{C} \subseteq \sqrt{n} \mathcal{E}_P^*.$$

Conjecture (Proved for aligned axes or $m = 2$): $\mathcal{C}$ intersection of m ellipsoids (joint eccentricity κ , $l = \min(n, m)$):

$$\mathcal{E}_P^* \subseteq \mathcal{C} \subseteq \sqrt{\frac{l\kappa^2}{l-1+\kappa^2}} \mathcal{E}_P^*.$$

If $\kappa \rightarrow \infty$, then

$$\mathcal{E}_P^* \subseteq \mathcal{C} \subseteq \sqrt{\min(n, m)} \mathcal{E}_P^*.$$



Linear inclusions and switched systems

Matrices $A_1, \dots, A_m \in \mathbb{R}^{n \times n}$.

Switched system

$$x_k = A_{\sigma_k} x_{k-1}, \quad \sigma_k \in [m]$$

$$\begin{aligned} \rho(\mathcal{A}) &= \min_{S \subseteq \mathbb{R}^n} \gamma \\ \text{subject to: } & \frac{A_{\sigma}}{\gamma} S \subseteq S, \quad i \in [m]. \end{aligned}$$

$\rho_{\text{SOS-}2d}(\mathcal{A})$: for polyset S of degree $2d$.

Guarantee

$$\begin{aligned} \rho(\mathcal{A}) &\leq \rho_{\text{SOS-}2d}(\mathcal{A}) \leq \alpha_{2d} \rho(\mathcal{A}) \\ \alpha_2 &= \sqrt{\min(n, m)} \\ \alpha_{2d} &= \min \left(\binom{n+d-1}{d}, m \right)^{\frac{1}{2d}}. \end{aligned}$$

Labelled graph $G(V, E)$.

Constrained switched system

$$x_k = A_{\sigma_k} x_{k-1}, (u_{k-1}, u_k, \sigma_k) \in E$$

$$\begin{aligned} \rho(G, \mathcal{A}) &= \min_{S_u \subseteq \mathbb{R}^n, u \in V} \gamma \\ \text{subject to: } & \frac{A_{\sigma}}{\gamma} S_u \subseteq S_v, (u, v, \sigma) \in E. \end{aligned}$$

$\rho_{\text{SOS-}2d}(G, \mathcal{A})$: for polysets S_u of degree $2d$.

Guarantee

$$\begin{aligned} \rho(G, \mathcal{A}) &\leq \rho_{\text{SOS-}2d}(G, \mathcal{A}) \leq \alpha_{2d} \rho(G, \mathcal{A}) \\ \alpha_2 &= \sqrt{\min(n, 2^{h(E)})} \\ \alpha_{2d} &= \min \left(\binom{n+d-1}{d}, 2^{h(E)} \right)^{\frac{1}{2d}}. \end{aligned}$$

Infeasibility certificate μ_σ for polysets $\rightarrow$ invalid for generic set program.

Rounding procedure generates **backward** switching sequence:

If **atomic**, take x_0 :

$$\sigma_{k+1} = \arg \max_{\sigma} \mu_{\sigma}(\{x_k / \|x_k\|_2\}), \quad x_{k+1} = A_{\sigma_{k+1}} x_k$$

Guarantee

$$\lim_{k \rightarrow \infty} \|A_{\sigma_1} \cdots A_{\sigma_k}\|^{\frac{1}{k}} = \rho(G, \mathcal{A}) = \rho_{\text{SOS-2d}}(G, \mathcal{A}).$$

Else, take $p_0(x)$ SOS:

$$\sigma_{k+1} = \arg \max_{\sigma} \langle \mu_{\sigma}, p_k(x) \rangle, \quad p_{k+1}(x) = p_k(A_{\sigma_{k+1}} x)$$

Guarantee

$$2^{-\frac{h(E)}{2d}} \rho_{\text{SOS-2d}}(G, \mathcal{A}) \leq \lim_{k \rightarrow \infty} \|A_{\sigma_1} \cdots A_{\sigma_k}\|^{\frac{1}{k}} \leq \rho(G, \mathcal{A}) \leq \rho_{\text{SOS-2d}}(G, \mathcal{A}).$$

Geometric duality

Discrete-time

$$CS_i \subseteq ES_j.$$

Inequality with support function or polar:

$$\begin{aligned}\forall z \in \mathbb{R}^r, \quad \delta^*(C^\top z | S_i) &\leq \delta^*(E^\top z | S_j) \\ \forall z \in \mathbb{R}^r, \quad g(S_i^\circ, C^\top z) &\leq g(S_j^\circ, E^\top z).\end{aligned}$$

Continuous time

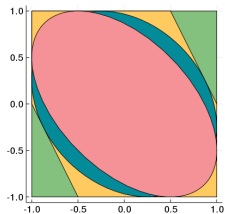
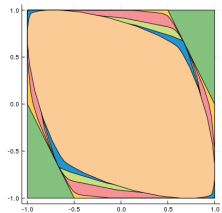
$$\forall x \in \partial S_i, Cx \in ET_{S_j}(x).$$

Inequality with support function or polar:

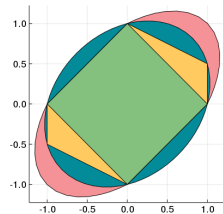
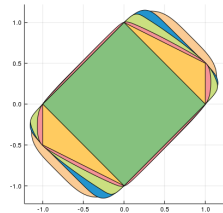
$$\begin{aligned}\forall z \in \mathbb{R}^r, \langle z, C \nabla \delta^*(E^\top z | S_i) \rangle &\leq 0 \\ \forall z \in \mathbb{R}^r, \langle z, C \nabla g(S_i^\circ, E^\top z) \rangle &\leq 0.\end{aligned}$$

Examples

Primal space



Dual space



Discrete-time

$$x_{k+1} = Ax_k + Bu_k.$$

Controlled invariance of $\mathcal{S}$:

$$\forall x \in \mathcal{S}, \exists u, Ax + Bu \in \mathcal{S}.$$

Equivalent to

$$\pi_{\text{Im}(B)^\perp} \mathcal{S} \subseteq \pi_{\text{Im}(B)^\perp} A\mathcal{S}.$$

Continuous time

$$\dot{x}(t) = Ax(t) + Bu(t).$$

Controlled invariance of $\mathcal{S}$:

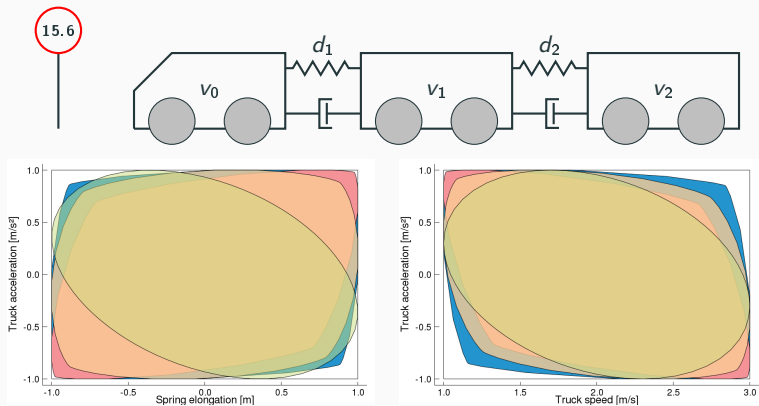
$$\forall x \in \partial\mathcal{S}, \exists u, Ax + Bu \in T_{\mathcal{S}}(x).$$

Equivalent to

$$\forall x \in \partial\mathcal{S}, \pi_{\text{Im}(B)^\perp} Ax \in \pi_{\text{Im}(B)^\perp} T_{\mathcal{S}}(x).$$

Applications

Model Predictive Control



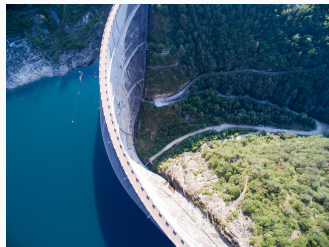
No relatively complete recourse:

- feasibility cuts or
- penalization.

→ Slow down Stochastic Dual Dynamic Programming.

Feasibility part of stochastic program → hybrid control algebraic system.

Controlled invariant set ensures relatively complete recourse.



Invariant under inner- and self-adhesivity.

$$\text{ia}_{J,K|I}(h) = \{g \in \mathcal{E}_{n'} \mid \forall I \subseteq L \subseteq J \cap K, g_L = h_{L \cap J} + h_{L \cap K} - h_I\}, \quad I = J \cap K,$$

$$\text{sa}_{J|I}(h) = \{g \in \mathcal{E}_{n'} \mid \forall I \subseteq L \subseteq J' \cap K, g_L = h_{L \cap J'} + h_{L \cap K} - h_I\}, \quad I \subseteq J,$$

Generate feasibility cuts with Dual Dynamic Programming.



Conclusion

- Set programs allow to model many important problems.
- **Guarantees** for templates can be developed for subclasses.
- Infeasibility certificate for a template may be **rounded**.
- Geometric **similarities** of different templates.
- **Geometric duality** enable reformulation in functional inequality.

