

# Geometric control of algebraic systems

ADHS 2021

---

Benoît Legat, Raphaël Jungers

July, 2021

## Algebraic approach: uncontrolled system

Linear system with  $A \in \mathbb{R}^{n \times n}$ :

$$\dot{x} = Ax.$$

Ellipsoid

$$\mathcal{E}_P = \{x \in \mathbb{R}^n \mid x^\top P x \leq 1\}.$$

Nagumo condition:

$$x^\top P A x \leq 0, \forall x \in \mathbb{R}^n.$$

LMI:

$$A^\top P + P A \leq 0.$$

## Algebraic approach: control system

Linear control system with  $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times n_u}$ :

$$\dot{x} = Ax + Bu.$$

Nagumo condition:

$$\forall x \in \mathbb{R}^{n_x}, \exists u \in \mathbb{R}^{n_u}, x^\top P(Ax + Bu) \leq 0$$

Linear state feedback  $u(x) = Kx$ ,  $\dot{x} = (A + BK)x$ , BMI:

$$A^\top P + PA + K^\top B^\top P + PBK \leq 0.$$

Change of variables  $Q := P^{-1}$  and  $Y := KQ$ , LMI

$$QA^\top + AQ + Y^\top B^\top + BY \leq 0.$$

## Algebraic approach: higher degree (Prajna et al. 2004)

$$\text{Polyset } \{x \mid x_{[d]}^\top P x_{[d]} \leq 1\}, \quad u(x) = K_1(x)x + K_2(x)x_{[d]},$$

$$\dot{x} = Ax + BK_1(x)x + BK_2(x)x_{[d]}$$

$$x_{[d]}^\top PM(x)BK_2(x)x_{[d]} + x_{[d]}^\top PF(x)x$$

$$\begin{bmatrix} x_{[d]} \\ x \end{bmatrix}^\top \begin{bmatrix} PM(x)BK_2(x) & PF(x) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{[d]} \\ x \end{bmatrix}.$$

$$\forall x \in \mathbb{R}^n, \quad \begin{bmatrix} 2PM(x)BK_2(x) & PF(x) \\ F^\top(x)P & 0 \end{bmatrix} \leq 0$$

$$\forall x \in \mathbb{R}^n, \quad \begin{bmatrix} 2M(x)BY(x) & F(x) \\ F^\top(x) & 0 \end{bmatrix} \leq 0$$

## Example

Linear system  $\dot{x} = -x$  and invariant set  $\{x \mid x^4 \leq 1\}$ . This holds

$$\begin{bmatrix} x^2 \\ x \end{bmatrix}^T \begin{bmatrix} 0 & -2x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x^2 \\ x \end{bmatrix} = -x^4$$

But this matrix is indefinite for  $x \neq 0$ :

$$\begin{bmatrix} 0 & -2x \\ -2x & 0 \end{bmatrix}$$

## Continuous time

$$\dot{x}(t) = Ax(t) + Bu(t).$$

$$\forall x \in \partial\mathcal{S}, \exists u, Ax + Bu \in T_{\mathcal{S}}(x).$$

$$\forall x \in \partial\mathcal{S}, \pi_{\text{Im}(B)^\perp} Ax \in \pi_{\text{Im}(B)^\perp} T_{\mathcal{S}}(x).$$

# Continuous-time controlled invariance<sup>1</sup>

$$\forall x \in \partial\mathcal{S}, Cx \in ET_{\mathcal{S}}(x).$$

Inequality with support function or polar:

$$\forall z \in \mathbb{R}^r, \langle z, C\nabla\delta^*(E^{\top}z|\mathcal{S}) \rangle \leq 0$$

$$\forall z \in \mathbb{R}^r, \langle z, C\nabla g(\mathcal{S}^{\circ}, E^{\top}z) \rangle \leq 0.$$

## Proof

$$\forall x \in \partial\mathcal{S}, \forall z \in (ET_{\mathcal{S}}(x))^{\circ}, \langle z, Cx \rangle \leq 0$$

$$\forall x \in \partial\mathcal{S}, \forall z \in E^{-\top}N_{\mathcal{S}}(x), \langle z, Cx \rangle \leq 0 \quad T_{\mathcal{S}}(x)^{\circ} = N_{\mathcal{S}}(x)$$

$$\forall z \in \mathbb{R}^r, \forall x \in F_{\mathcal{S}}(E^{\top}z), \langle z, Cx \rangle \leq 0 \quad x \in F_{\mathcal{C}}(y) \Leftrightarrow y \in N_{\mathcal{C}}(x)$$

$$\forall z \in \mathbb{R}^r, \langle z, C\nabla\delta^*(E^{\top}z|\mathcal{S}) \rangle \leq 0.$$

---

<sup>1</sup>L. and Jungers, *Geometric control of algebraic systems*, ADHS 2021

## Example

Control system with  $x \in [-1, 1]^2$ ,  $u \in [-1, 1]$ :

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

Control system with  $x \in [-1, 1]^3$ ,  $u$  unconstrained:

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u.$$

Algebraic system with  $x \in [-1, 1]^3$ :

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x$$

# Templates

## Gauge or Minkowski function

$$g(\mathcal{S}, x) \triangleq \min_{\gamma} \{ \gamma : x \in \gamma \mathcal{S}, \gamma \geq 0 \}$$

## Face fan or polytope

Face fan of polytope  $\mathcal{S}$  is  $\mathcal{P}_i$  s.t.

$$g(\mathcal{S}, x) = \max_{i \in [m]} \langle a_i, x \rangle = \langle a_i, x \rangle \text{ if } x \in \mathcal{P}_i$$

$$g(\mathcal{S}, x) = \langle a_i, x \rangle \text{ if } x \in \mathcal{P}_i$$

polytope of fixed face fan

$$g(\mathcal{S}, x) = \sqrt{x^\top Q x}$$

ellipsoid

$$g(\mathcal{S}, x) = \sqrt{x^\top Q_i x} \text{ if } x \in \mathcal{P}_i$$

piecewise semi-ellipsoid

$$g(\mathcal{S}, x) = \sqrt[2d]{p(x)}$$

poly(nomial sublevel )set

# Solutions

