

Piecewise semi-ellipsoidal control invariant sets

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Gauge function

$$\mathcal{S} = \{x \mid g(\mathcal{S}, x) \leq 1\}.$$

Inclusion:

$$\mathcal{S} \subseteq \mathcal{T} \Leftrightarrow \forall x, g(\mathcal{S}, x) \geq g(\mathcal{T}, x).$$

Linear **preimage**:

$$g(A^{-1}\mathcal{S}, x) = g(\mathcal{S}, Ax).$$

Ellipsoid: $g(\mathcal{S}, x) = \sqrt{x^\top Q x}.$

Autonomous system

Classical LMI for stability:

$$x_{k+1} = Ax_k$$

$$A\mathcal{S} \subseteq \mathcal{S}$$

$$\mathcal{S} \subseteq A^{-1}\mathcal{S}$$

$$g(\mathcal{S}, x) \geq g(A^{-1}\mathcal{S}, x)$$

$$g(\mathcal{S}, x) \geq g(\mathcal{S}, Ax)$$

$$Q \succeq A^\top Q A.$$

Control invariant set: Algebraic approach

Discrete-time control system

$$x_{k+1} = Ax_k + Bu_k.$$

Controlled invariance of $\mathcal{S}$:

$$\forall x \in \mathcal{S}, \exists u, Ax + Bu \in \mathcal{S}.$$

Linear state feedback

$u(x) = Kx \rightarrow$ autonomous system:

$$x_{k+1} = (A + BK)x_k.$$

From previous slide:

$$Q \succeq (A + BK)^T Q (A + BK).$$

Cubic matrix inequality!

Schur complement

$$\begin{bmatrix} Q & (A + BK)^T Q \\ Q(A + BK) & Q \end{bmatrix} \succeq 0.$$

Bilinear matrix inequality!

Change of variables

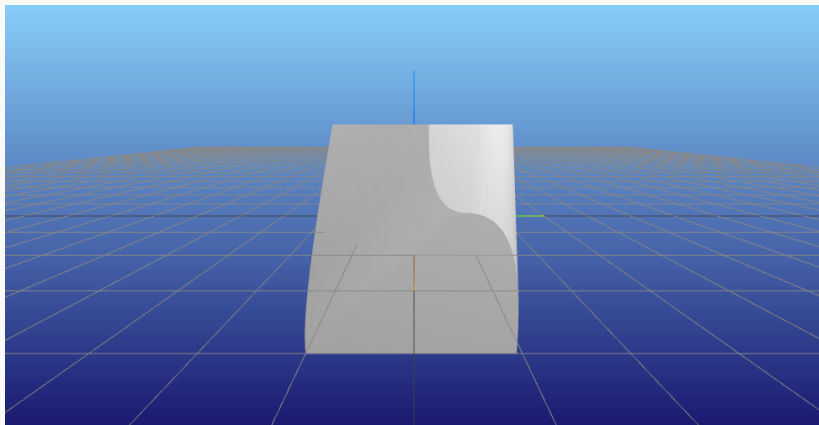
Q^{-1} and $Y := KQ^{-1}$.

$$\begin{bmatrix} Q^{-1} & Q^{-1}A^T + Y^T B^T \\ AQ^{-1} + BY & Q^{-1} \end{bmatrix} \succeq 0.$$

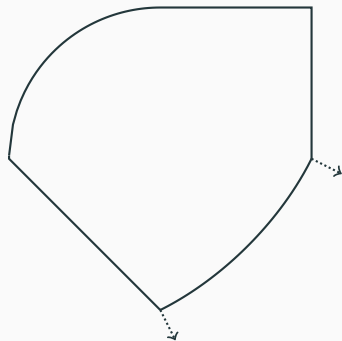
Linear matrix inequality!

How to generalize this for other sets ?

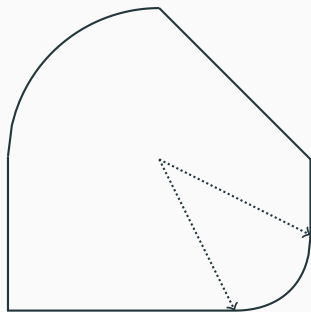
Need for piecewise



Piecewise semi-ellipsoids and polarity



$$\begin{cases} x^2 & \text{if } 0 \leq y \leq x, \\ y^2 & \text{if } 0 \leq x \leq y, \\ x^2 + y^2 & \text{if } x \leq 0 \leq y, \\ (x + y)^2 & \text{if } x, y \leq 0, \\ x^2 - xy + y^2 & \text{if } y \leq 0 \leq x. \end{cases}$$



$$\begin{cases} (x + y)^2 & \text{if } 0 \leq x, y, \\ x^2 + y^2 & \text{if } x \leq 0 \leq y, \\ x^2 & \text{if } x \leq y \leq 0, \\ y^2 & \text{if } y \leq x, 2x \leq -y, \\ \frac{4}{3}(x^2 + xy + y^2) & \text{if } 2x \geq -y, x \leq -2y, \\ x^2 & \text{if } x \geq -2y, y \geq 0. \end{cases}$$

Control invariance and algebraic invariance through support function

Support function

$$\begin{aligned}h(\mathcal{S}, y) &= \sup_{x \in \mathcal{S}} y^\top x \\ &= g(\mathcal{S}^\circ, x)\end{aligned}$$

Inclusion:

$$\mathcal{S} \subseteq \mathcal{T} \Leftrightarrow \forall y, h(\mathcal{S}, y) \leq h(\mathcal{T}, y).$$

Linear **image**: $h(A\mathcal{S}, y) = h(\mathcal{S}, A^\top y)$.

Ellipsoid:

$$\begin{aligned}g(\mathcal{S}, x) &= \sqrt{x^\top Q x} \\ h(\mathcal{S}, y) &= \sqrt{y^\top Q^{-1} y}\end{aligned}$$

Control system

$$x_{k+1} = Ax_k + Bu_k.$$

Controlled invariance of $\mathcal{S}$:

$$\forall x \in \mathcal{S}, \exists u, Ax + Bu \in \mathcal{S}.$$

Equivalent to

$$\begin{aligned}\pi_{\text{Im}(B)^\perp} \mathcal{S} &\subseteq \pi_{\text{Im}(B)^\perp} A\mathcal{S} \\ h(\mathcal{S}, \pi_{\text{Im}(B)^\perp}^\top y) &\leq h(\mathcal{S}, A^\top \pi_{\text{Im}(B)^\perp}^\top y)\end{aligned}$$

Algebraic system

$$\pi_{\text{Im}(B)^\perp} x_{k+1} = \pi_{\text{Im}(B)^\perp} Ax_k.$$

$$\pi_{\text{Im}(B)^\perp} Q^{-1} \pi_{\text{Im}(B)^\perp}^\top \preceq \pi_{\text{Im}(B)^\perp} A Q^{-1} A^\top \pi_{\text{Im}(B)^\perp}^\top$$

Piecewise semi-ellipsoid

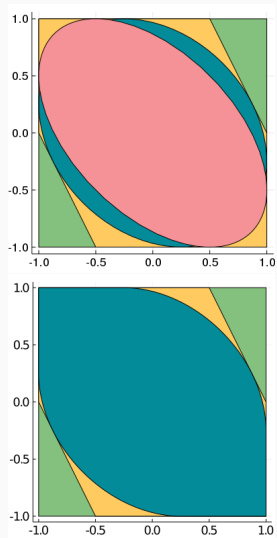
Fixed polyhedral conic partition $\mathcal{P}_i$:

$$h(\mathcal{S}, x) = \sqrt{x^\top Q_i x} \quad \text{if } x \in \mathcal{P}_i$$

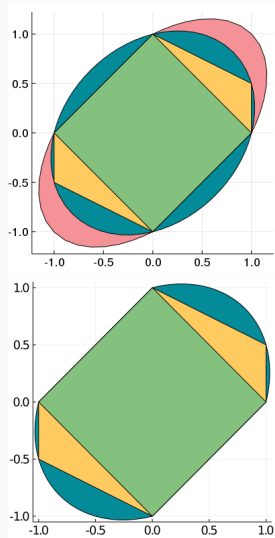
$$\begin{aligned} & \text{find } Q_i \succeq 0, u_{i,j} \in \mathbb{R}^n \quad \text{such that} \\ & x^\top C Q_i C^\top x \leq x^\top E Q_j E^\top x, \forall x \in C^{-\top} \mathcal{P}_i \cap E^{-\top} \mathcal{P}_j, \\ & \quad \forall i, j \in \{1, \dots, m\} \\ & v^\top Q_i v \leq 1, \quad \forall v \in \mathcal{V}_{\text{rep}}(\mathcal{X}^\circ) \cap \mathcal{P}_i, \forall i \\ & Q_i - Q_j = u_{i,j} n_{ij}^\top + n_{ij} u_{i,j}^\top, \quad \forall i, j \in \mathcal{N} \\ & Q_j n_{ij} - Q_i n_{ij} \in (\mathcal{P}_i \cap \mathcal{P}_j)^\circ, \quad \forall i, j \in \mathcal{N}. \end{aligned}$$

Example

Primal space



Dual space



- Ellipsoids are **cheap** to compute but conservative.
- Polyhedra representation are **expressive** but can have huge size in some regions.
- Piecewise semi-ellipsoids combine **advantages** of both.

Future work:

- How to obtain conic partition ?
 - A few backward propagation of a polytope.
 - Refine with dual solution.
- Piecewise polysets (polynomial support function).