

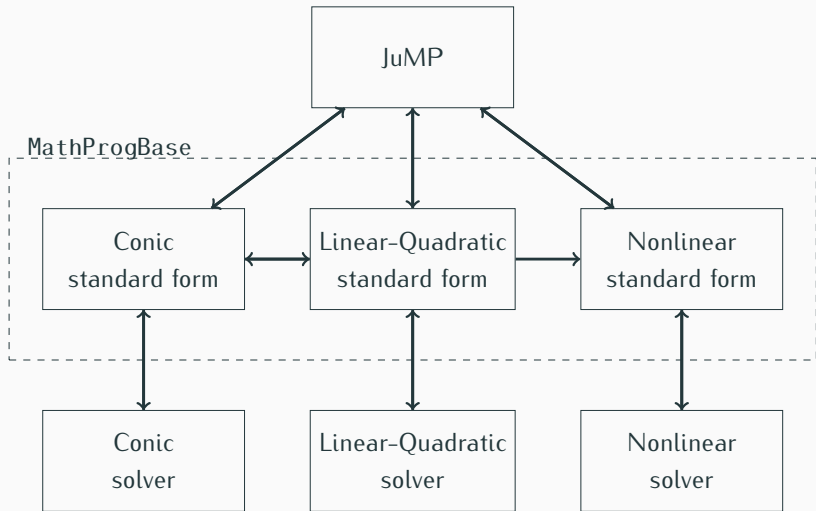
MathOptInterface: A Data Structure for Mathematical Optimization Problems

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JuMP – as it was



We propose to formulate problems as:

$$\begin{array}{ll}\min_{x \in \mathbb{R}^N} & f_0(x) \\ \text{subject to:} & f_i(x) \in S_i,\end{array}$$

where $f_i \in \mathcal{F}$ and $S_i \in \mathcal{S}$.

Examples of $\mathcal{F}$

- SingleVariable: $x \rightarrow x_i$
- VectorOfVariables: $x \rightarrow [x_i, x_j, \dots, x_k]$
- ScalarAffineFunction: $x \rightarrow a^\top x + b$
- VectorAffineFunction: $x \rightarrow Ax + b$
- ScalarQuadraticFunction, VectorQuadraticFunction, ScalarNonlinearFunction, ...

Examples of scalar-valued $\mathcal{S}$

- LessThan: $(-\infty, u] \subseteq \mathbb{R}$
- GreaterThan: $[l, \infty) \subseteq \mathbb{R}$
- Interval: $[l, u] \subseteq \mathbb{R}$
- EqualTo: $\{v\} \subseteq \mathbb{R}$
- Integer: $\mathbb{Z} \subseteq \mathbb{R}$
- ZeroOne: $\{0, 1\} \subseteq \mathbb{R}$
- Semicontinuous, Semiinteger, ...

Examples of vector-valued $\mathcal{S}$

- Nonnegatives: $\{x \in \mathbb{R}^N : x \geq 0\} \subseteq \mathbb{R}^N$
- Nonpositives: $\{x \in \mathbb{R}^N : x \leq 0\} \subseteq \mathbb{R}^N$
- Zeros: $\{0\} \subseteq \mathbb{R}^N$
- SecondOrderCone: $\{(t, x) \in \mathbb{R}^N : \|x\|_2 \leq t\} \subseteq \mathbb{R}^N$
- PositiveSemidefiniteConeTriangle, SOS1, SOS2, Complements, ...

Multiplicity of formulations

$l \leq a^\top x \leq u$ can be formulated in at least three ways:

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(ScalarAffineFunction-in-LessThan) and $a^\top x \geq l$
(ScalarAffineFunction-in-GreaterThan)
3. Introducing a slack variable y , with constraints $a^\top x - y = 0$
(ScalarAffineFunction-in-EqualTo) and $l \leq y \leq u$
(SingleVariable-in-Interval)

Bridging example

$$\begin{array}{ll}\min & x^2 + x + 1 \\ \text{s.t.} & x \in \mathbb{R}_+^1.\end{array}$$

to

- VectorAffineFunction-in-RotatedSecondOrderCone constraints;
- ScalarAffineFunction objective functions; and
- Variables in Nonnegatives.

Objective slack bridge

$$\begin{array}{ll}\min & x^2 + x + 1 \\ \text{s.t.} & x \in \mathbb{R}_+^1.\end{array}$$

becomes

$$\begin{array}{ll}\min & y \\ \text{s.t.} & x^2 + x + 1 \leq y \\ & x \in \mathbb{R}_+^1 \\ & y \text{ free.}\end{array}$$

Objective functionize bridge

$$\begin{array}{ll}\min & y \\ \text{s.t.} & x^2 + x + 1 \leq y \\ & x \in \mathbb{R}_+^1 \\ & y \text{ free.}\end{array}$$

becomes

$$\begin{array}{ll}\min & 1y + 0 \\ \text{s.t.} & x^2 + x + 1 \leq y \\ & x \in \mathbb{R}_+^1 \\ & y \text{ free.}\end{array}$$

Constraint quadratic to RSOC bridge

Let $Q = U^T U$, we have

$$0.5x^T Qx + a^T x + b \leq 0$$

$$\iff \|Ux\|_2 \leq 2(-a^T x - b)$$

$$\iff [1, -a^T x - b, Ux] \in \text{RotatedSecondOrderCone}(3),$$

$$\min \quad 1y + 0$$

$$\text{s.t.} \quad x^2 + x + 1 \leq y$$

$$x \in \mathbb{R}_+^1$$

$$y \text{ free}$$

becomes

$$\min \quad 1y + 0$$

$$\text{s.t.} \quad [1, -x + y - 1, \sqrt{2}x] \in \text{RotatedSecondOrderCone}(3)$$

$$x \in \mathbb{R}_+^1$$

$$y \text{ free.}$$

Variable free bridge

$$\begin{aligned} \min \quad & 1y + 0 \\ \text{s.t.} \quad & [1, -x + y - 1, \sqrt{2}x] \in \text{RotatedSecondOrderCone}(3) \\ & x \in \mathbb{R}_+^1 \\ & y \text{ free.} \end{aligned}$$

becomes

$$\begin{aligned} \min \quad & 1y^+ - 1y^- + 0 \\ \text{s.t.} \quad & [1, -x + 1y^+ - 1y^- - 1, \sqrt{2}x] \in \text{RotatedSecondOrderCone}(3) \\ & [x, y^+, y^-] \in \mathbb{R}_+^3. \end{aligned}$$

FlipSignBridge

- Variable $x \geq l$ substituted by $x = -y$ where $y \leq -l$.
- Constraint $a^\top x \leq \beta$ transformed into $-a^\top x \geq -\beta$.

VectorizeBridge

- Variable $x \geq l$ substituted by $x = y + l$ where $y \in \mathbb{R}_+^1$.
- Constraint $a^\top x \leq \beta$ transformed into $[a^\top x - \beta] \in \mathbb{R}_-^1$.

IntervalBridge

- Constraint $l \leq a^\top x \leq u$ transformed into $a^\top x \geq l$ and $a^\top x \leq u$.

IndicatorBridge

- **Constraint** y binary, $\neg y \Rightarrow a^\top x \leq \beta$ transformed into z binary, $z + y = 1$ and $z \Rightarrow a^\top x \leq \beta$.

FreeBridge

- **Variable** $x \in \mathbb{R}$ substituted by $x = y + z$ where $y \in \mathbb{R}_+$ and $z \in \mathbb{R}_-$.

SlackBridge

- **Constraint** $f \in S$ transformed into $f = x$ for variable $x \in S$.
- **Objective** $\min f$ transformed into $\min x$ with $f \leq x$.

Bridge hyper-graph

- Consider directed hyper-graph $G(N, E)$ with one node $n \in N$ for each function-in-set pair.
- Solver declares set of supported nodes S .
- Each directed hyper-edge $e \in E$, is a bridge, and a source node $s(e) \in N$ and a set of target nodes $T(e) \subseteq N$.
- Each edge has weight $w(e)$

$$C(n) = \begin{cases} 0 & n \in S \\ \min_{e \in E \mid s(e)=n} \{w(e) + \sum_{n' \in T(e)} C(n')\} & \text{otherwise.} \end{cases}$$

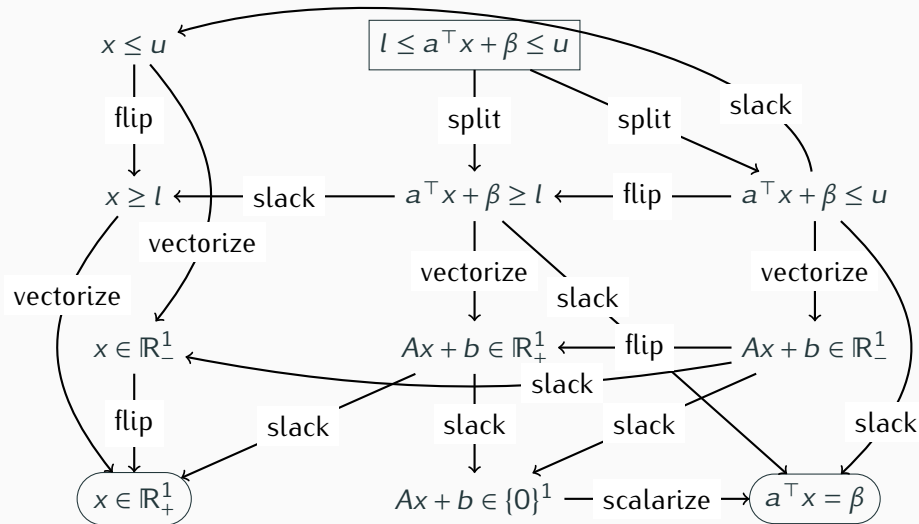
Hyper-graph nodes

There are actually three types of nodes

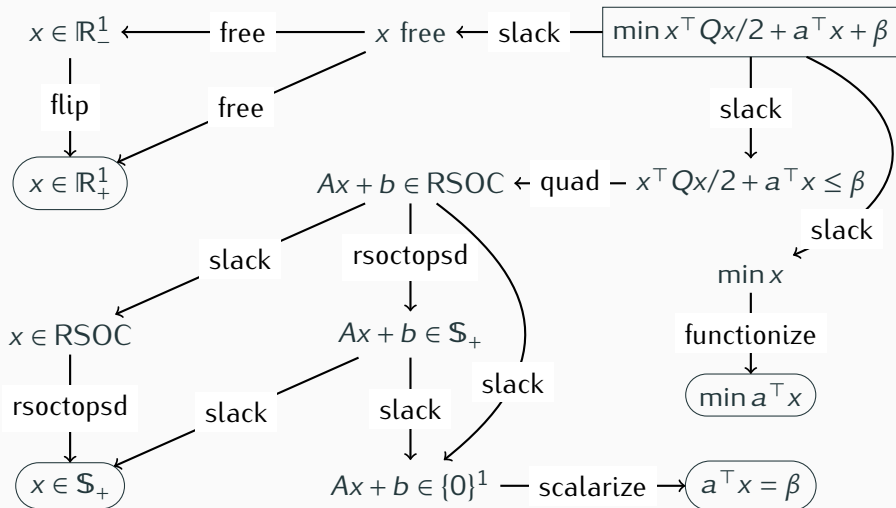
- Objective nodes, for each objective $\min f$
- Constraint nodes, for each f -in- S
- Variable nodes for each variable-in- S

Hyper-edges start at a node, and can target nodes of different types.

Interval constraint for SDPA



Quadratic objective for SDPA



JuMP and MathOptInterface provide an optimization framework **extensible** by **design**.

Bridges allow to communicate **custom** structure to solvers from **solver-independent** models.