

Computing polynomial controlled invariant sets using Sum-of-Squares Programming

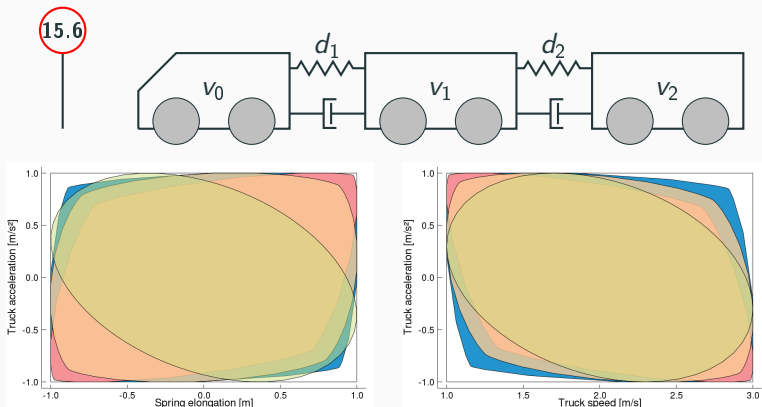
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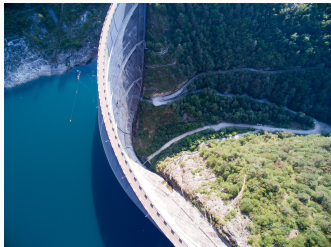
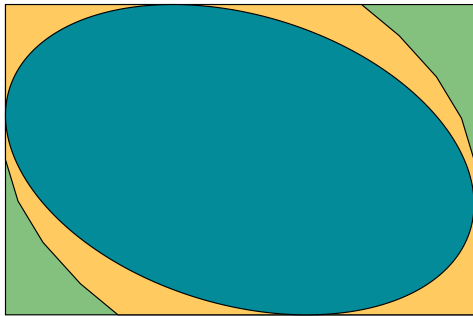
See presentation at *37rd Symposium on Information Theory in the Benelux*, 2016.

Safe Model Predictive Control



See presentation at *IFAC Conference on Analysis and Design of Hybrid Systems (ADHS)*, 2018.

Safe Stochastic Programming



See presentation at *23rd International Symposium on Mathematical Programming (ISMP)*, 2018.

Running example

Consider control system:

$$v_{k+1} = v_k + a_k \Delta t$$

$$a_{k+1} = u_k$$

The system is $x_{k+1} = Ax_k + Bu_k$ where

$$A = \begin{bmatrix} 1 & \Delta t \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

As we will see, a set S is controlled invariant iff

$$\begin{bmatrix} 1 & \Delta t \end{bmatrix} S \subseteq \begin{bmatrix} 1 & 0 \end{bmatrix} S$$

Linear discrete control system

$$x_{k+1} = Ax_k + Bu_k, \quad x_k \in X, u_k \in U$$

Controlled invariant set

Set $S \subseteq X$ *controlled invariant* iff

$$\forall x \in S, \exists u \in U \text{ such that } Ax + Bu \in S$$

Unconstrained input without loss of generality

Constrained input

$$x_{k+1} = Ax_k + Bu_k, \quad x_k \in X, u_k \in U$$

Lift to unconstrained input: $x' = (x, u)$

$$x_{k+1} = \begin{pmatrix} A & B \end{pmatrix} x'_k$$

$$u_{k+1} = u'_k$$

where $x'_k \in X' = X \times U$.

Unconstrained control to algebraic system

Linear Discrete **Control** System (DT**CS**)

$$x_{k+1} = Ax_k + Bu_k, \quad x_k \in X$$

Linear Discrete **Algebraic** System (DT**AS**)

$$Ex_{k+1} = Ax_k, \quad x_k \in X$$

Invariant Set $S \subseteq X$ if

$$\forall x \in S, \exists y \in S : Ey = Ax$$

Reduction

The set S is controlled invariant for the DT**CS** iff it is invariant for the DT**AS**

$$\Pi x_{k+1} = \Pi Ax_k, \quad x_k \in X$$

where Π is a projection on $\text{Im}(B)^\perp$.

Ellipsoidal controlled invariant Set of Algebraic Systems

$$Ex_{k+1} = Ax_k, \quad x_k \in X$$

Convex set S invariant iff

$$AS \subseteq ES$$

$$(AS)^\circ \supseteq (ES)^\circ$$

$$A^{-T}S^\circ \supseteq E^{-T}S^\circ$$

Ellipsoidal computation

$$C^\circ = \{x \mid x^T Q x \leq 1\} \quad \Rightarrow \quad A^{-T}C^\circ = \{x \mid x^T AQA^T x \leq 1\}$$

S-procedure:

$$EQE^T - AQA^T \text{ is PSD.}$$

Example with ellipsoids: Model

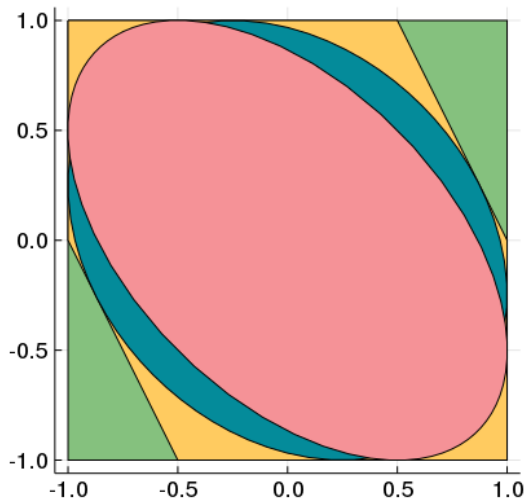
Maximal volume ellipsoid (determinant):

```
model = Model(...)
@variable(model, S, Ellipsoid(symmetric=true))
@constraint(model, S ⊆ □)
@constraint(model, A * S ⊆ E * S)
@objective(model, Max, nth_root(volume(S)))
@time JuMP.optimize!(model)
ell = □ JuMP.value(S)
```

Maximal sum of the squares of the semi-axes of the ellipsoid (trace):

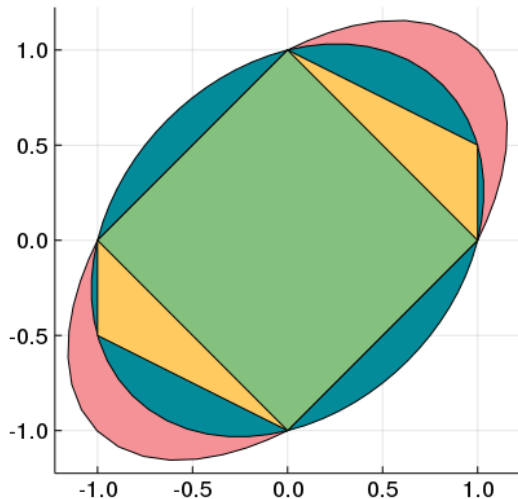
```
@objective(model, Max,
            L1_heuristic(volume(S), [1.0, 1.0]))
```

Example with ellipsoids: Primal Solution



Maximal volume (blue) and maximal sum of the squares of the semi-axes (red).

Example with ellipsoids: Polar Solution



Maximal volume (blue) and maximal sum of the squares of the semi-axes (red).

Support function vs polar set

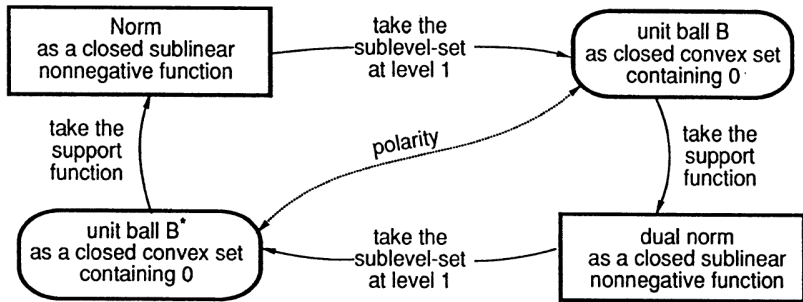


Fig. 3.2.1. Dual norms and polar sets

Hiriart-Urruty, Lemaréchal, *Fundamentals of convex analysis*, 2001.

Support function interpretation

Support function:

$$\delta^*(y|\mathcal{C}) = \sup_{x \in \mathcal{C}} \langle y, x \rangle.$$

Key property:

$$\delta^*(y|AC) = \delta^*(A^\top y|\mathcal{C}).$$

$$AS \subseteq ES$$

$$\delta^*(y|AS) \leq \delta^*(y|ES) \quad \forall y$$

$$\delta^*(A^\top y|S) \leq \delta^*(E^\top y|S) \quad \forall y$$

Ellipsoidal computation

$$\delta^*(y|\mathcal{C}) = \sqrt{x^\top Qx}$$

Polynomial support function

Polynomial $p(x)$, e.g. $p(x) = x^4 + y^4$.

Polar interpretation

$$C^\circ = \{x \mid p(x) \leq 1\}.$$

Support function interpretation

$$\delta^*(y|C) = p(x)^{\frac{1}{2d}}.$$

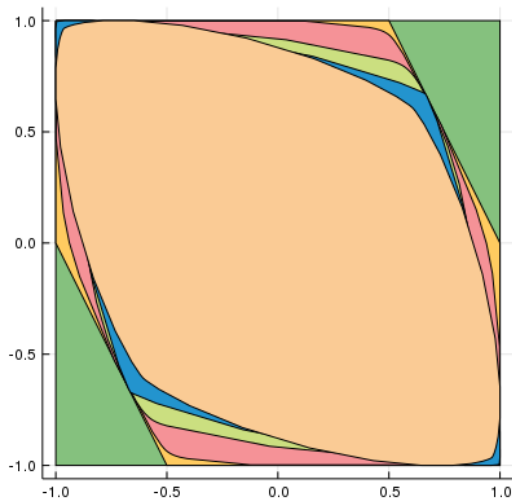
Example with polynomial sublevel set: Model

```
@variable(model, S, PolySet(degree=d, convex=true,  
                             symmetric=true))
```

Use L1 norm as volume heuristic.

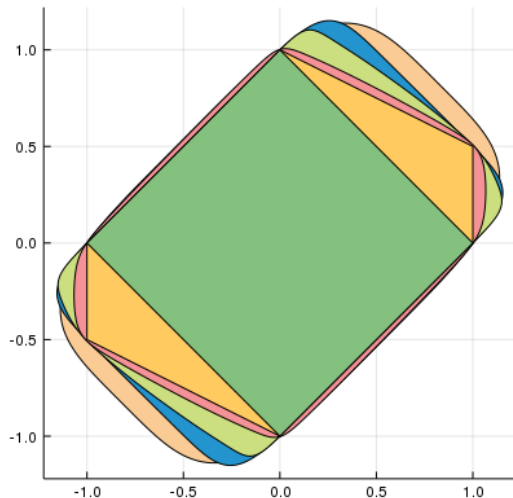
Convexity ensured by Sum-of-Squares convexity, i.e., $y^\top \text{Hess}(x)y$ is Sum-of-Squares for all x, y .

Example with polynomial sublevel set: Primal Solution



Degree 4 (orange), 8 (blue), 18 (green) and 22 (red).

Example with polynomial sublevel set: Polar Solution

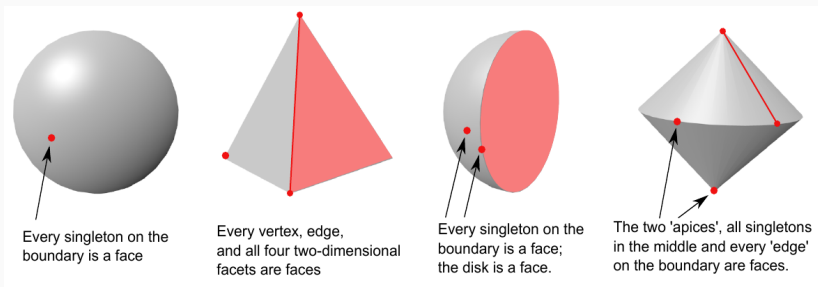


Degree 4 (orange), 8 (blue), 18 (green) and 22 (red).

Exposed face

Exposed face of convex set $\mathcal{C}$ associated to y is:

$$F_{\mathcal{C}}(y) = \{x \in \mathcal{C} \mid \langle x, y \rangle = \delta^*(y|\mathcal{C})\}.$$



Vera Roshchina, *Faces of convex sets*.

Continuous-time **control** system:

$$\dot{x} = Ax + Bu \quad (1)$$

Continuous-time **algebraic** system, $E, A \in \mathbb{R}^{r \times n}$:

$$E\dot{x} = Ax. \quad (2)$$

Convex set invariant iff:

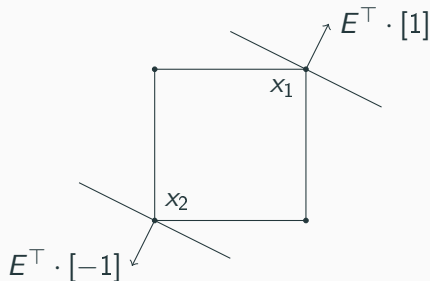
$$\forall z \in \mathbb{R}^r, \forall x \in F_C(E^\top z), \langle z, Ax \rangle \leq 0.$$

Polyhedral controlled invariant set

Suppose $r = 1$, $F_C(E^\top \cdot [-1]) = \{x_1\}$, $F_C(E^\top \cdot [1]) = \{x_2\}$,
Square invariant iff

$$Ax_1 \leq 0$$

$$Ax_2 \geq 0.$$



Polynomial support function

If $\delta^*(\cdot|\mathcal{C})$ is differentiable:

$$\forall z \in \mathbb{R}^r, \langle z, A \nabla \delta^*(E^\top z | \mathcal{C}) \rangle \leq 0.$$

If $\delta^*(y|\mathcal{C}) = \sqrt{y^\top P y}$, $\mathcal{C}$ controlled invariant iff

$$AP^{-1}E^\top + EP^{-1}A^\top \preceq 0.$$

If $\delta^*(y|\mathcal{C}) = p(y)^{\frac{1}{2d}}$, then

$$\nabla \delta^*(y|\mathcal{C}) = \frac{1}{p(y)^{1-\frac{1}{2d}}} \nabla p(y)$$

and $\mathcal{C}$ controlled invariant iff

$$z^\top A \nabla p(E^\top z) \leq 0 \quad \forall z \in \mathbb{R}^r.$$

Conclusion

- Controlled invariance condition based on **support function**.
- **Semidefinite** program for quadratic support function.
- **Sum-of-Squares** program for polynomial support function of arbitrary degree.
- Compute invariant sets with the `SetProg.jl` and `SumOfSquares.jl` extensions of `JuMP.jl` (stickers available)

