

Computing semialgebraic invariant sets for dynamical systems

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Set program

Set inclusion

Consider two sets $S \subseteq \mathbb{R}^n$, $T \subseteq \mathbb{R}^m$, matrices

$A \in \mathbb{R}^{r \times n}$, $B \in \mathbb{R}^{r \times m}$, and vector $c \in \mathbb{R}^r$:

$$AS + c \subseteq BT.$$

Set program

Given fixed sets T_i , find sets S_i :

$$\max_{S_i} f(\text{vol}(S_1), \dots, \text{vol}(S_n))$$

$$A_j S_{a_j} + c_j \subseteq B_j S_{b_j}$$

$$S_i \subseteq T_i$$

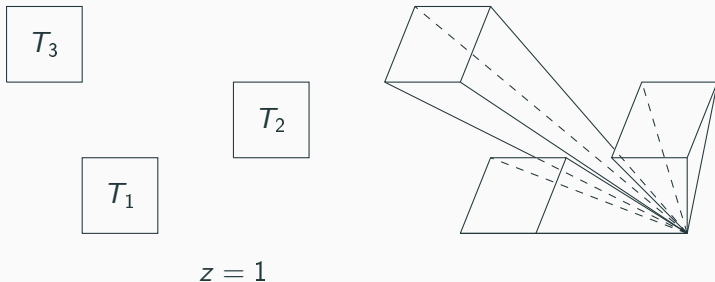
Homogeneous if T_i are symmetric around the origin and $c_j = 0$.

Homogenization

Non-homogeneous polynomial : $x^4y^2 + x^2y^4 + 1 - 3x^2y^2$.

Homogenization : $x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$.

If T_i have different centers: Homogenization with new variable $z = 1$.



We now look for **cones** K instead of **set** S .

We replace **polar** S° with **dual** K^* .

Set computation

Pre-image: $A^{-1}S = \{x \mid Ax \in S\}$, A can be non-singular or rectangular.

Ellipsoid

$$S_i = \{x \mid x^\top Q_i x \leq 1\}$$

$$A^{-1}S_i = \{x \mid x^\top A^\top Q_i A x \leq 1\}$$

$$S_i \subseteq S_j \Leftrightarrow Q_i \succeq Q_j.$$

Polynomial sublevel set

$$S_i = \{x \mid p_i(x) \leq 1\}$$

$$A^{-1}S_i = \{x \mid p_i(Ax) \leq 1\}$$

$$S_i \subseteq S_j \Leftrightarrow p_i(x) \geq p_j(x).$$

Set inclusions

Simple inclusion

$$AS \subseteq T$$

we rewrite into

$$S \subseteq A^{-1}T$$

Polar trick

Inclusion with two linear maps:

$$AS \subseteq BT \not\Rightarrow B^{-1}S \subseteq A^{-1}T$$

If S and T are convex, it is equivalent to

$$\begin{aligned}(AS)^\circ &\supseteq (BT)^\circ \\ A^{-\top}S^\circ &\supseteq B^{-\top}T^\circ\end{aligned}$$

Controlled invariant set

Linear discrete control system

$$x_{k+1} = Ax_k + Bu_k, \quad x_k \in X, u_k \in U$$

Controlled invariant set

Set $S \subseteq X$ *controlled invariant* iff

$$\forall x \in S, \exists u \in U \text{ such that } Ax + Bu \in S$$

Safe control

- Compute controlled invariant set $S \subseteq X$.
- Replace X by S .

Unconstrained input without loss of generality

Constrained input

$$x_{k+1} = Ax_k + Bu_k, \quad x_k \in X, u_k \in U$$

Lift to unconstrained input: $x' = (x, u)$

$$x_{k+1} = \begin{pmatrix} A & B \end{pmatrix} x'_k$$

$$u_{k+1} = u'_k$$

where $x'_k \in X' = X \times U$.

Unconstrained control to algebraic system

Linear Discrete **Control** System (DT**CS**)

$$x_{k+1} = Ax_k + Bu_k, \quad x_k \in X$$

Linear Discrete **Algebraic** System (DT**AS**)

$$Ex_{k+1} = Ax_k, \quad x_k \in X$$

Invariant Set $S \subseteq X$ if

$$\forall x \in S, \exists y \in S : Ey = Ax$$

Reduction

The set S is controlled invariant for the DT**CS** iff it is invariant for the DT**AS**

$$\Pi x_{k+1} = \Pi Ax_k, \quad x_k \in X$$

where Π is a projection on $\text{Im}(B)^\perp$.

Computing controlled invariant sets

$$\max_S \text{vol}(S)$$

$$\Pi AS \subseteq \Pi S$$

$$S \subseteq T$$

Polar trick:

$$\min_{S^\circ} \text{vol}(S^\circ)$$

$$(\Pi A)^{-\top} S^\circ \supseteq \Pi^{-\top} S^\circ$$

$$S^\circ \supseteq T^\circ$$

Suppose $T = \{x : |\langle h_i, x \rangle| \leq 1\}$, using ellipsoids

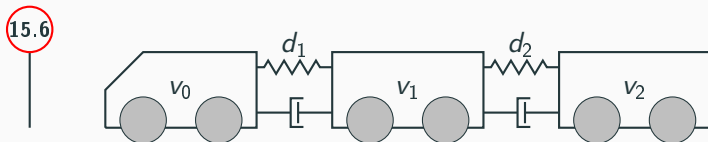
$$\min_{Q \succ 0} -\log \det(Q)$$

$$\Pi A Q A^\top \Pi^\top \preceq \Pi Q \Pi^\top$$

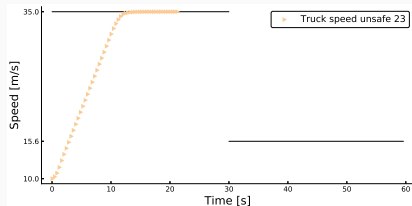
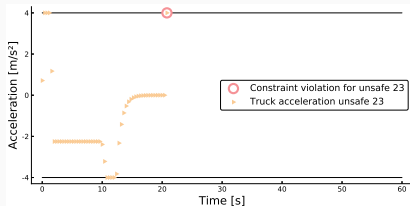
$$h_i^\top Q h_i \leq 1 \quad \forall i$$

Safety in control requires **long** horizon

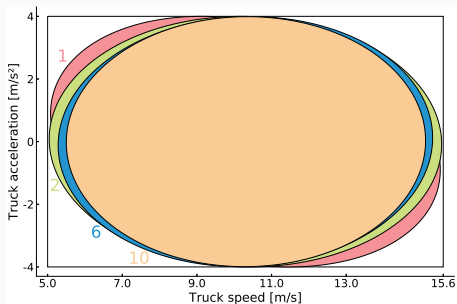
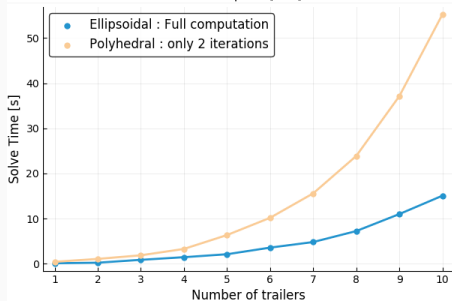
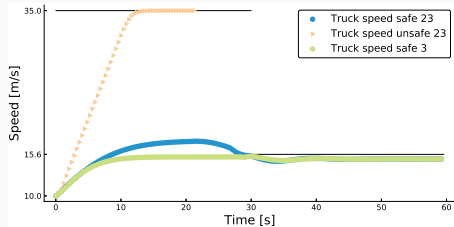
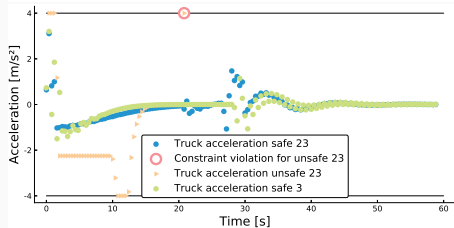
Speed limit changes from 35 m s^{-1} to 15.6 m s^{-1} after 30 s.



One trailer, step-size of 0.4 s, horizon of 9.2 s/23 step-sizes:



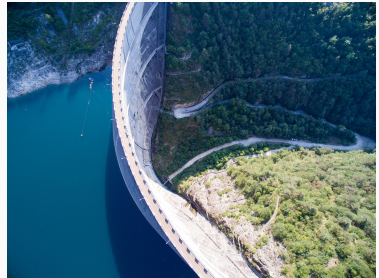
Controlled invariant set



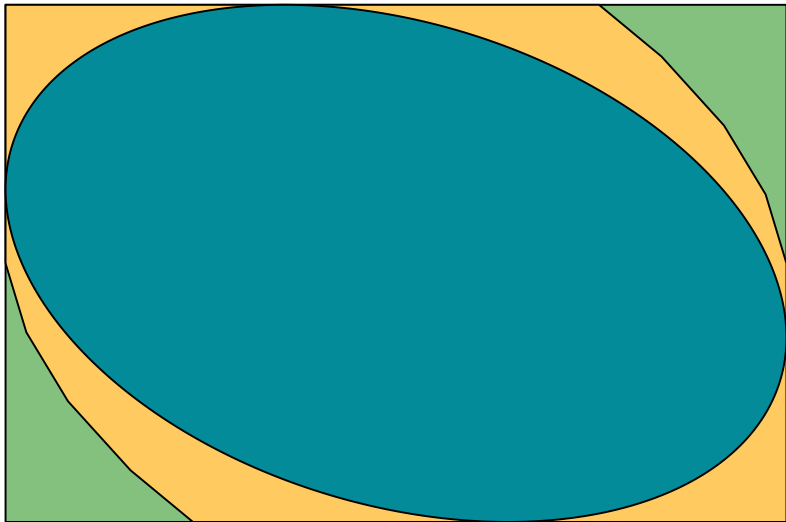
Water Reservoir

- $h_{\min} \leq h \leq h_{\max}$: water height [m]
- $v_{\min} \leq v \leq v_{\max}$: height increase rate [m s^{-1}]
- $u_{\min} \leq u \leq u_{\max}$: v increase rate [m s^{-2}]

- $h > h_{\max}$: Water spills, ecological damage
- $h < h_{\min}$: Risk of electricity shortage



Controlled invariant set



Switched Systems

$$x_{k+1} = A_{\sigma_k} x_k, \quad \sigma_k \in \{1, \dots, m\}.$$

Switched System **stable** $\iff \lim_{k \rightarrow \infty} x_k = 0 \ \forall x_0.$

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Joint Spectral Radius (JSR) $\mathcal{A} = \{A_1, \dots, A_m\}$

$$\rho(\mathcal{A}) := \limsup_{k \rightarrow \infty} \max_{\sigma \in \{1, \dots, m\}} \|A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}\|^{1/k}.$$

$$\text{Switched System stable} \iff \rho(\mathcal{A}) < 1$$

Joint Spectral Radius and Switched Systems

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Switched System **stable** $\iff \rho(\mathcal{A}) < 1$

- If $m = 1$, “standard” spectral radius $\rho(A_1) = \max_i |\lambda_i|$.
- If $m \geq 2$, much more complicated...
 - Determining if $\rho(\mathcal{A}) < 1$ is **NP-hard** (Tsitsiklis & Blondel 1997).
 - Determining if $\rho(\mathcal{A}) \leq 1$ is **undecidable** (Blondel & Tsitsiklis 2000).

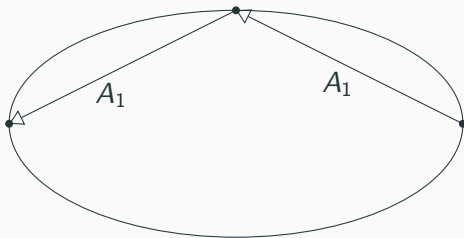
Joint Spectral Radius and Invariant Sets

Invariant Set

$$A_{\sigma}S \in S, \sigma \in \{1, \dots, m\}$$

Example

$$m = 1, A_1 = \text{Rotation}(90^\circ) \times \begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix}$$



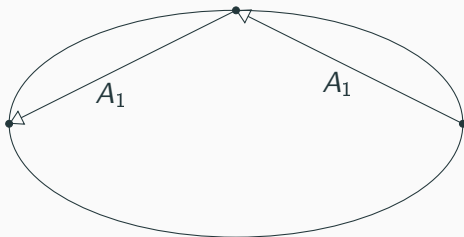
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$$\exists \text{ invariant set for } A_1/\gamma, \dots, A_m/\gamma \implies \rho(\mathcal{A}) \leq \gamma$$

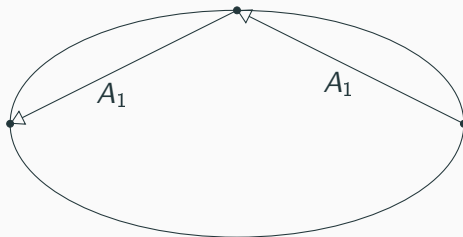
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SOS- $2d$ invariant sets

Can compute invariant sets using Sum of Squares Programming (SOS) : $S = \{x : p(x) \leq 1\}$ where $p(x)$ homogeneous SOS degree $2d$.

P.A. Parrilo and A. Jadbabaie, Approximation of the joint spectral radius using sum of squares, *Linear Algebra and its Applications*, Vol. 428, No. 10, pp. 2385–2402, 2008.

SOS-2d invariant sets

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Guarantee

$$\rho_{\text{SOS-2d}}(\mathcal{A}) = \inf_{\text{s.t. } \exists \text{ SOS-2d invariant set for } A_1/\gamma, \dots, A_m/\gamma} \gamma$$

$$\eta^{-\frac{1}{2d}} \rho_{\text{SOS-2d}}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \rho_{\text{SOS-2d}}(\mathcal{A})$$

where $\eta = \min \left\{ m, \binom{n+d-1}{d} \right\}$ and $A_\sigma \in \mathbb{R}^{n \times n}$.

As $d \rightarrow \infty$, $\eta^{-\frac{1}{2d}} \rightarrow 0$.

P.A. Parrilo and A. Jadbabaie, Approximation of the joint spectral radius using sum of squares, *Linear Algebra and its Applications*, Vol. 428, No. 10, pp. 2385–2402, 2008.

Measures

Measures μ defined on the sphere

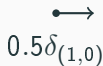
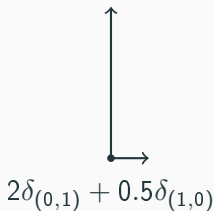
$$\mathbb{S}^{n-1} = \{x : \|x\|_2 = 1\}$$

For $B \in \mathbb{S}^{n-1}$, $\mu(B)$ is **measure** of B .

atomic $\Leftrightarrow$ sum of **Diracs**

amplitude in drawing represents

- **mass** of atom for *atomic* measure,
- **density** otherwise.



Transfer/Frobenius-Perron operator and push-forward measure

Usual definition:

$$(A\#\mu)(B) = \mu(A^{-1}(B))$$

Problem: $\mathbb{S}^{n-1}$ not invariant under A_σ .

Solution: Alternative definition:

$$\int_{\mathbb{S}^{n-1}} f(x)(A\#\mu)(dx) = \int_{\mathbb{S}^{n-1}} f(Ax)\mu(dx) \quad \forall f.$$

Example

$$A_1 = \text{Rotation}(90^\circ) \times \begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix}$$



Periodic trajectory

Periodic trajectory of growth rate γ :

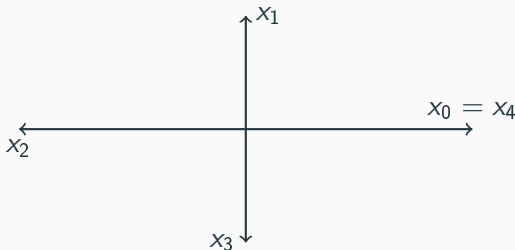
$$\gamma^k x_0 = A^k x_0$$

$\exists x_i, 1 \leq i \leq k$, such that:

$$\gamma x_{i+1} = Ax_i$$

Example

$$A = \text{Rotation}(90^\circ) \times \begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix}, x_0 = (1, 0), k = 4$$



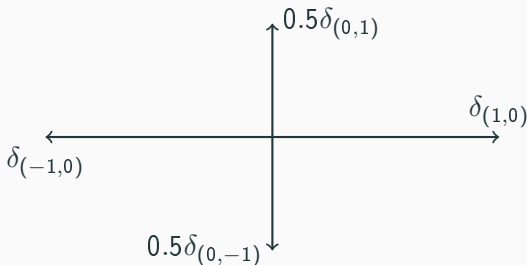
Periodic trajectory as atomic measure I

Equivalent to:

$$\gamma\delta_{x_{k+1}} = A\#\delta_{x_k}$$

Example

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Periodic trajectory as atomic measure II

Convexification: Find μ_k such that for all $k = 1, \dots$

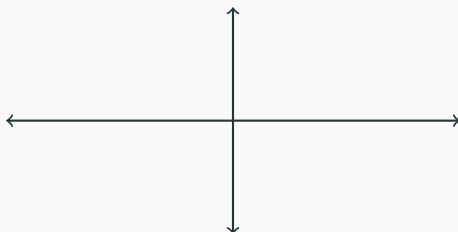
$$\gamma\mu_{k+1} = A\#\mu_k.$$

Project the time:

$$\gamma\mu = A\#\mu.$$

Example

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$$\mu = A\#\mu$$

Periodic trajectory as atomic measure III

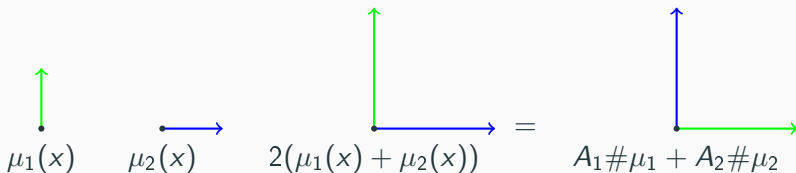
$$\gamma(\mu_1 + \mu_2) = A_1 \# \mu_1 + A_2 \# \mu_2.$$

periodic trajectory of growth rate $\gamma \leftrightarrow$ atomic measures.

Example

$$A_1 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$



Theorem

Let μ_σ , $\sigma = 1, \dots, m$, be measures on $\mathbb{S}^{n-1}$, that satisfies

$$\sum_{\sigma=1}^m A_\sigma \# \mu_\sigma \geq \gamma \sum_{\sigma=1}^m \mu_\sigma,$$

and at least one of the expectations is nonzero.

Then $\rho(\mathcal{A}) \geq \gamma$.

SOS- $2d$ invariant measures

Moments $\tilde{\mu}$ of degree $2d$. Can write constraint

$$\langle \tilde{\mu}, p(x) \rangle = \int_{\mathbb{S}^{n-1}} p(x) d\tilde{\mu} \geq 0, \quad \forall p(x) \text{ SOS of degree } 2d.$$

as Semidefinite Programming constraint.

Dual to invariant sets program.

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- Can detect when moments of **atomic** measures and recover atoms by solving an algebraic system. The moments are computed numerically with errors around 10^{-6} , currently doing Gröbner basis computation in double precision floating point numbers in Julia. We should try HomotopyContinuation.jl.
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pseudo-measure

$$\langle \tilde{\mu}, p(x) \rangle \geq 0, \quad \forall p(x) \text{ SOS.}$$

Making a liar tell the truth

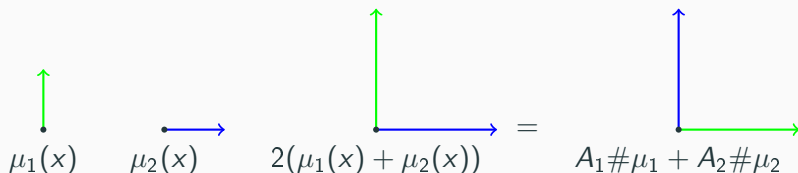
If **atomic**, take x_0 :

$$\sigma_{k+1} = \arg \max_{\sigma} \mu_{\sigma}(\{x_k / \|x_k\|_2\}), \quad x_{k+1} = A_{\sigma_{k+1}} x_k$$

Example

$$A_1 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$



If **pseudo-measure**, take $p_0(x)$ SOS:

$$\sigma_{k+1} = \arg \max_{\sigma} \langle \mu_{\sigma}, p_k(x) \rangle, \quad p_{k+1}(x) = p_k(A_{\sigma_{k+1}} x)$$

Note: Switching built **backward**!

Infinite backward sequence

Guarantee

$$m^{-\frac{1}{2d}} \rho_{\text{SOS-2d}}(\mathcal{A}) \leq \lim_{k \rightarrow \infty} \|A_{\sigma_1} \cdots A_{\sigma_k}\|^{\frac{1}{k}} \leq \rho(\mathcal{A}) \leq \rho_{\text{SOS-2d}}(\mathcal{A}).$$

If sequence **periodic** $\rightarrow$ can compute growth rate

B. L., R.M. Jungers and P.A. Parrilo, Generating unstable trajectories for Switched Systems via Dual Sum-Of-Squares techniques, *Proceedings of the 19th International Conference on Hybrid Systems: Computation and Control*, ACM, pp. 51–60, 2016.

Constraint Joint Spectral Radius

For set program with several S_i , consider the graph where nodes are S_i and edges are constraints $A_{\sigma_j} S_{a_j} \subseteq S_{b_j}$

Allowed switching sequences $\sigma_1, \sigma_2, \dots \leftrightarrow$ paths of graph.

$$\rho(A)^{-\frac{1}{2d}} \rho_{\text{SOS-2d}}(G, \mathcal{A}) \leq \lim_{k \rightarrow \infty} \|A_{\sigma_1} \cdots A_{\sigma_k}\|^{\frac{1}{k}} \leq \rho(G, \mathcal{A}) \leq \rho_{\text{SOS-2d}}(G, \mathcal{A}).$$

where A is adjacency matrix of the graph.

Conclusion

- Set programs can be solved using sum of squares programming
- This can be used to compute controlled invariant sets.
- With simple inclusion and homogeneity, infeasibility can be certified using the dual.