Computing controlled invariant sets using semidefinite programming

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Safe Model Predictive Control

Motivating application : Model Predictive Control

A truck with M trailers linked by mass-spring dampers.

Initial state

- Speed of truck and trailers $v_i = 10\,\mathrm{m\,s^{-1}}$.
- Spring elongation $d_i = 0\,\mathrm{m}$.

Objective

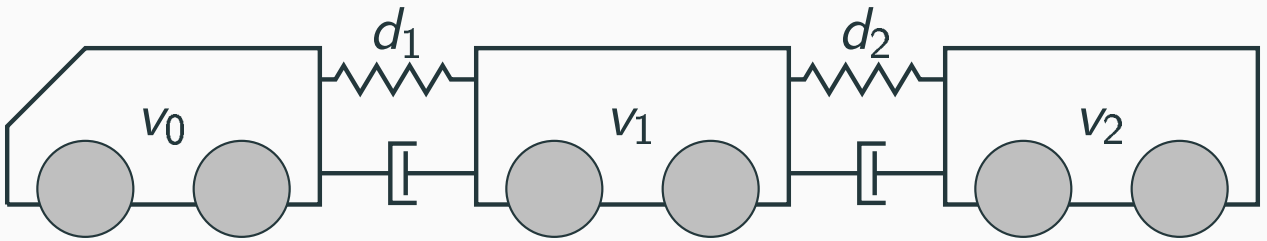
Maximize distance covered in 60 s.

Control

Acceleration of truck $|u| \leq 4\,\mathrm{m\,s^{-2}}$.

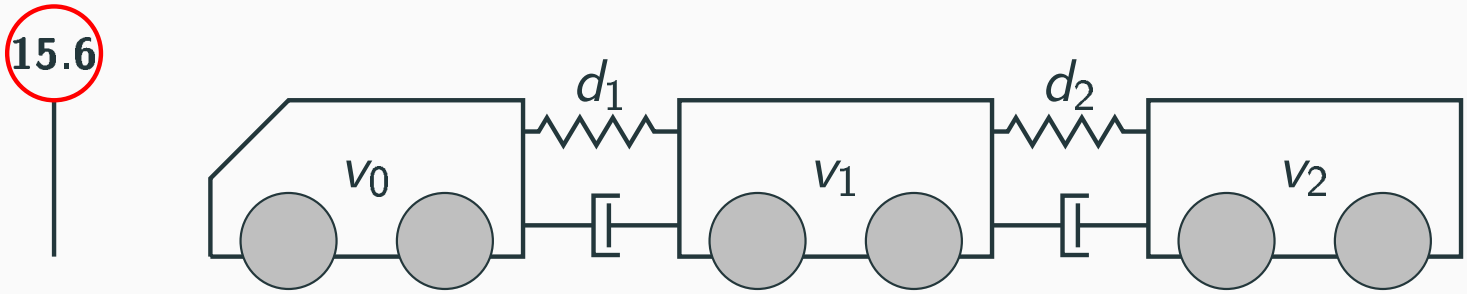
Safety requirements

- Speed of truck and each trailer $v_i \leq 35\,\mathrm{m\,s^{-1}}$.
- Spring elongation $|d_i| \leq 0.5\,\mathrm{m}$.

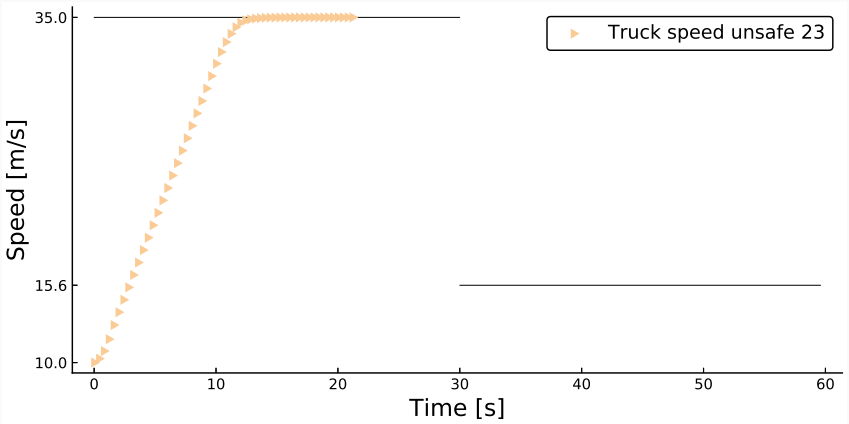


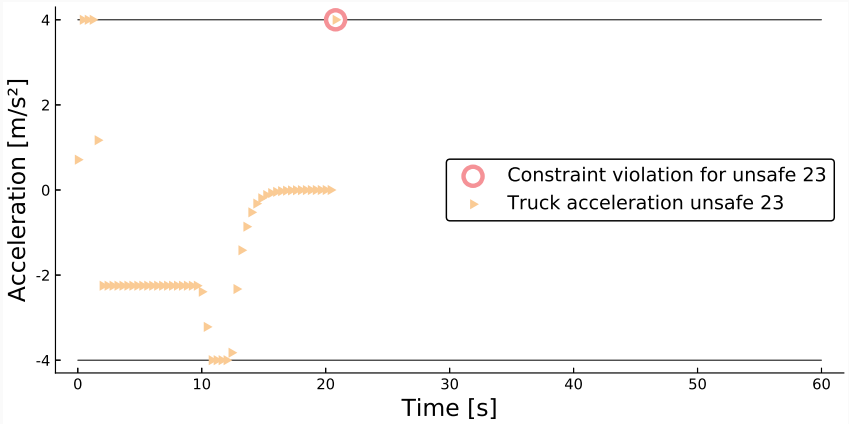
Safety requires long horizon

Speed limit changes from $35\,\mathrm{m\,s^{-1}}$ to $15.6\,\mathrm{m\,s^{-1}}$ after 30 s.



One trailer, step-size of 0.4 s, horizon of 9.2 s/23 step-sizes:





Controlled invariant set

Linear discrete control system

$$x_{k+1} = Ax_k + Bu_k, \quad x_k \in X, u_k \in U$$

Controlled invariant set
Set $S \subseteq X$ *controlled invariant* iff

$$\forall x \in S, \exists u \in U \text{ such that } Ax + Bu \in S$$

Safe MPC

- Compute controlled invariant set $S \subseteq$ safe set.
- Add constraint: state $\in S$.

Representation of polyhedral sets

$\mathcal{H}$-representation
Representation of polyhedron S as intersection of halfspaces:

$$S = \bigcap_{i=1}^{m} \{ x \mid a_i^\top x \leq \beta_i \}.$$

- Computing $A^{-1}S$ is easy: $a_i \to A^\top a_i$.
- Projection is hard. Algorithms: Fourier-Motzkin, ...

$\mathcal{V}$-representation
Representation of polyhedron S as hull of points/rays:

$$S = \text{convexhull}(x_1, \ldots, x_{m_x}) + \text{conichull}(r_1, \ldots, r_{m_r}).$$

- Computing AS is easy: $x \to Ax$, $r \to Ar$.
- Projection is easy: project each point/ray.

Representation conversion is hard. Algorithms: double description, quick hull, ...

Unconstrained input without loss of generality

Constrained input

$$x_{k+1} = Ax_k + Bu_k, \quad x_k \in X, u_k \in U$$

Lift to unconstrained input: $x' = (x, u)$

$$x_{k+1} = \begin{pmatrix} A & B \end{pmatrix} x'_k$$

$$u_{k+1} = u'_k$$

where $x'_k \in X' = X \times U$.

Computation of invariant set

Linear discrete system : $x_{k+1} = Ax_k,\ x_k \in X$.

Pre-image

$$A^{-1}S \triangleq \{\, x \mid Ax \in S \,\}$$

Invariant set

Set $S \subseteq X$ *invariant* iff

$$\forall x \in S, Ax \in S \quad \Leftrightarrow \quad AS \subseteq S \quad \Leftrightarrow \quad S \subseteq A^{-1}S$$

Computation of invariant set

Linear discrete system : $x_{k+1} = Ax_k$, $x_k \in X$.

Pre-image

$$A^{-1}S \triangleq \{ x \mid Ax \in S \}$$

Invariant set

Set $S \subseteq X$ *invariant* iff

$$\forall x \in S, Ax \in S \quad \Leftrightarrow \quad AS \subseteq S \quad \Leftrightarrow \quad S \subseteq A^{-1}S$$

Polyhedral computation

$$P_0 = X$$

$$P_{k+1} = P_k \cap A^{-1}P_k$$

A^{-1} easy with $\mathcal{H}$-representation

$\cap$ easy with $\mathcal{H}$-representation

Computation of invariant set

Linear discrete system : $x_{k+1} = Ax_k$, $x_k \in X$.

Pre-image

$$A^{-1}S \triangleq \{ x \mid Ax \in S \}$$

Invariant set

Set $S \subseteq X$ *invariant* iff

$$\forall x \in S, Ax \in S \quad \Leftrightarrow \quad AS \subseteq S \quad \Leftrightarrow \quad S \subseteq A^{-1}S$$

Ellipsoidal computation

$$C = \{ x \mid x^\top Qx \leq 1 \} \quad \Rightarrow \quad A^{-1}C = \{ x \mid x^\top A^\top QAx \leq 1 \}$$

S-procedure:

$$Q - A^\top QA \text{ is PSD.}$$

Polyhedral computation of controlled invariant set

$$
\begin{aligned}
P_0 &= X \\
P_{k+1} &= P_k \cap \text{proj}_x\{\,(x,u) \mid Ax + Bu \in P_k\,\} \\
&= P_k \cap \text{proj}_x [A,B]^{-1} P_k
\end{aligned}
$$

Not scalable...

$[A,B]^{-1}$	easy with $\mathcal{H}$-representation
proj_x	easy with $\mathcal{V}$-representation
$\cap$	easy with $\mathcal{H}$-representation

Unconstrained control to algebraic system

Linear Discrete Control System (DTCS)

$$x_{k+1} = Ax_k + Bu_k, \quad x_k \in X$$

Linear Discrete Algebraic System (DTAS)

$$Ex_{k+1} = Ax_k, \quad x_k \in X$$

Invariant Set $S \subseteq X$ if

$$\forall x \in S, \exists y \in S : Ey = Ax$$

Reduction

The set S is controlled invariant for the DTCS iff it is invariant for the DTAS

$$\Pi x_{k+1} = \Pi A x_k, \quad x_k \in X$$

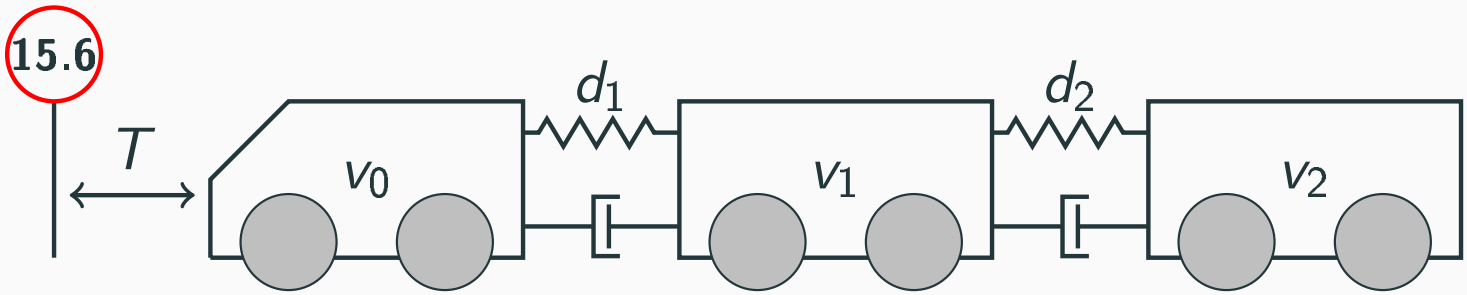
where Π is a projection on $\mathrm{Im}(B)^{\perp}$.

Generalization to Hybrid Systems

Back to our example

Four speed limits with a delay T to satisfy limit

$\rightarrow$ Hybrid System

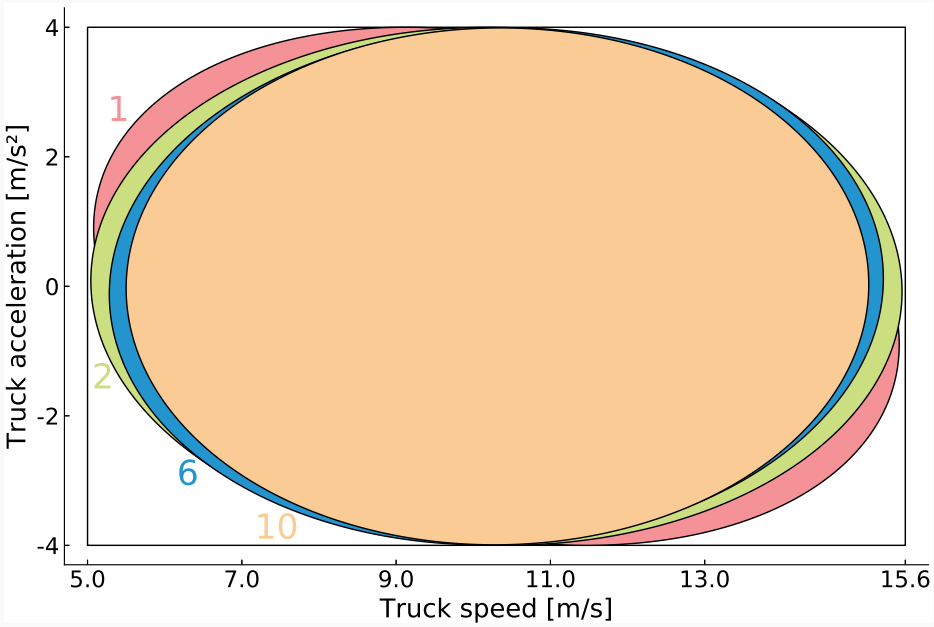


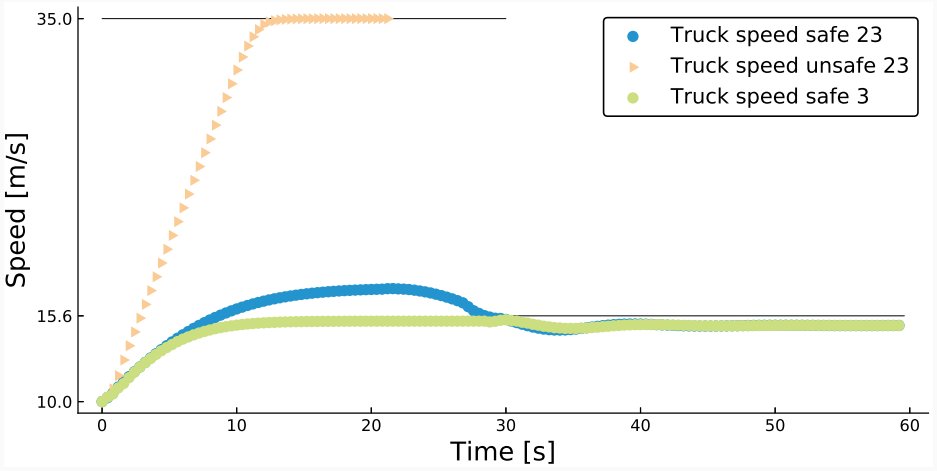
Generalization to Hybrid Systems:

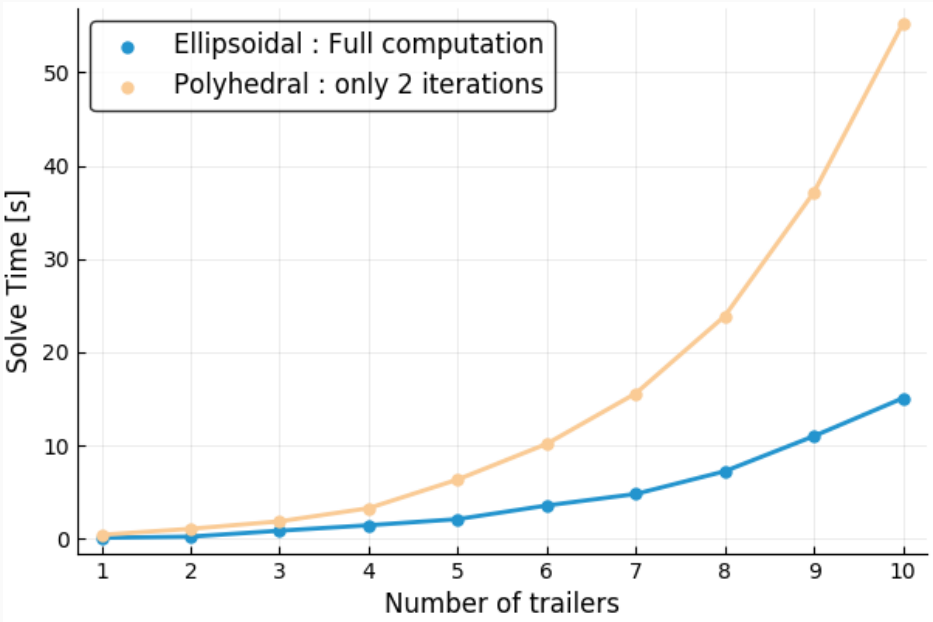
L., B.; Tabuada, P. & Jungers, R. M.
Computing controlled invariant sets for hybrid systems with applications to model-predictive control.

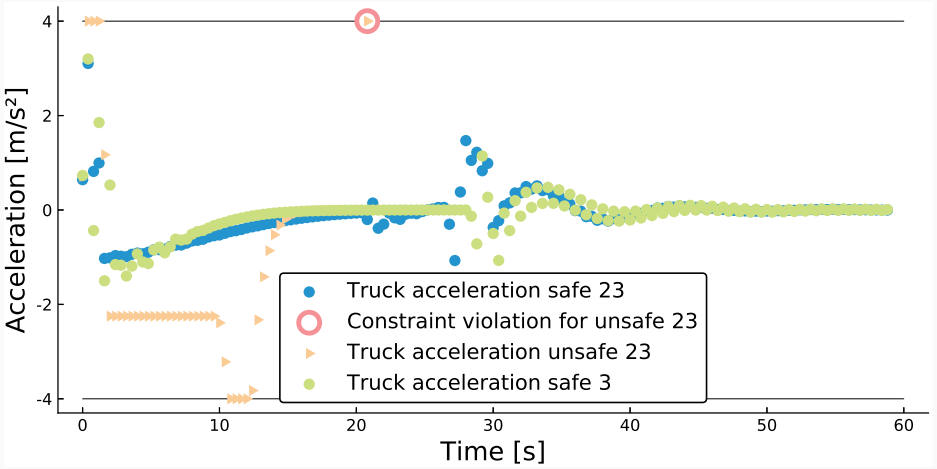
Proceedings of the IFAC Conference on Analysis and Design of Hybrid Systems, **2018**

Results


Truck acceleration [m/s²]
Truck speed [m/s]
1
2
6
10


Speed [m/s]
Time [s]
Truck speed safe 23
Truck speed unsafe 23
Truck speed safe 3


Solve Time [s]
Number of trailers
Ellipsoidal : Full computation
Polyhedral : only 2 iterations


Acceleration [m/s²]
Time [s]
Truck acceleration safe 23
Constraint violation for unsafe 23
Truck acceleration unsafe 23
Truck acceleration safe 3

Conclusion

Conclusion

- Safe MPC either needs long horizon or precomputed controlled invariant sets;
- Polyhedral computation of controlled invariant sets is computationally intensive;
- Ellipsoidal computation of controlled invariant sets is semidefinite representable in the dual space.