

Computing controlled invariant sets for hybrid systems with applications to model-predictive control

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Safe Model Predictive Control

Generalization to Hybrid Systems

Conclusion

Safe Model Predictive Control

Motivating application : Model Predictive Control

A truck with M trailers linked by mass-spring dampers.

Initial state

- Speed of truck and trailers $v_i = 10 \text{ m s}^{-1}$.
- Spring elongation $d_i = 0 \text{ m}$.

Objective

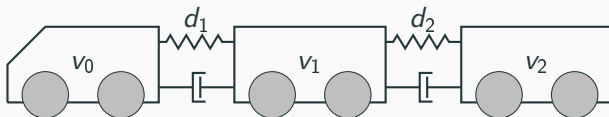
Maximize distance covered in 60 s.

Control

Acceleration of truck $|u| \leq 4 \text{ m s}^{-2}$.

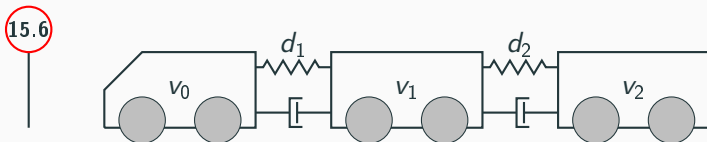
Safety requirements

- Speed of truck and each trailer $v_i \leq 35 \text{ m s}^{-1}$.
- Spring elongation $|d_i| \leq 0.5 \text{ m}$.

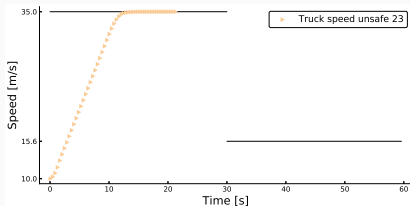
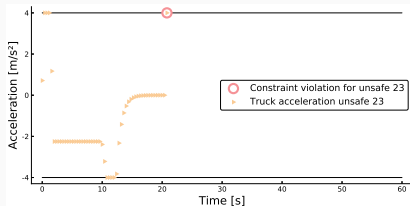


Safety requires long horizon

Speed limit changes from 35 m s^{-1} to 15.6 m s^{-1} after 30 s.



One trailer, step-size of 0.4 s, horizon of 9.2 s/23 step-sizes:



Controlled invariant set

Linear discrete control system

$$x_{k+1} = Ax_k + Bu_k, \quad x_k \in X, u_k \in U$$

Controlled invariant set

Set $S \subseteq X$ *controlled invariant* iff

$$\forall x \in S, \exists u \in U \text{ such that } Ax + Bu \in S$$

Safe MPC

- Compute controlled invariant set $S \subseteq$ safe set.
- Add constraint: state $\in S$.

Representation of polyhedral sets

$\mathcal{H}$ -representation

Representation of polyhedron S as intersection of halfspaces:

$$S = \bigcap_{i=1}^m \{x \mid a_i^\top x \leq \beta_i\}.$$

- Computing $A^{-1}S$ is **easy**: $a_i \rightarrow A^\top a_i$.
- Projection is **hard**. Algorithms: Fourier-Motzkin, ...

$\mathcal{V}$ -representation

Representation of polyhedron S as hull of points/rays:

$$S = \text{convexhull}(x_1, \dots, x_{m_x}) + \text{conichull}(r_1, \dots, r_{m_r}).$$

- Computing AS is **easy**: $x \rightarrow Ax$, $r \rightarrow Ar$.
- Projection is **easy**: project each point/ray.

Representation conversion is **hard**. Algorithms: double description, quick hull, ...

Unconstrained input without loss of generality

Constrained input

$$x_{k+1} = Ax_k + Bu_k, \quad x_k \in X, u_k \in U$$

Lift to unconstrained input: $x' = (x, u)$

$$x_{k+1} = \begin{pmatrix} A & B \end{pmatrix} x'_k$$

$$u_{k+1} = u'_k$$

where $x'_k \in X' = X \times U$.

Computation of invariant set

Linear discrete system : $x_{k+1} = Ax_k$, $x_k \in X$.

Pre-image

$$A^{-1}S \triangleq \{x \mid Ax \in S\}$$

Invariant set

Set $S \subseteq X$ *invariant* iff

$$\forall x \in S, Ax \in S \quad \Leftrightarrow \quad AS \subseteq S \quad \Leftrightarrow \quad S \subseteq A^{-1}S$$

Computation of invariant set

Linear discrete system : $x_{k+1} = Ax_k$, $x_k \in X$.

Pre-image

$$A^{-1}S \triangleq \{x \mid Ax \in S\}$$

Invariant set

Set $S \subseteq X$ *invariant* iff

$$\forall x \in S, Ax \in S \quad \Leftrightarrow \quad AS \subseteq S \quad \Leftrightarrow \quad S \subseteq A^{-1}S$$

Polyhedral computation

$$P_0 = X$$

$$P_{k+1} = P_k \cap A^{-1}P_k$$

A^{-1} easy with $\mathcal{H}$ -representation

$\cap$ easy with $\mathcal{H}$ -representation

Computation of invariant set

Linear discrete system : $x_{k+1} = Ax_k$, $x_k \in X$.

Pre-image

$$A^{-1}S \triangleq \{x \mid Ax \in S\}$$

Invariant set

Set $S \subseteq X$ *invariant* iff

$$\forall x \in S, Ax \in S \quad \Leftrightarrow \quad AS \subseteq S \quad \Leftrightarrow \quad S \subseteq A^{-1}S$$

Ellipsoidal computation

$$C = \{x \mid x^T Q x \leq 1\} \quad \Rightarrow \quad A^{-1}C = \{x \mid x^T A^T Q A x \leq 1\}$$

S-procedure:

$$Q - A^T Q A \text{ is PSD.}$$

$$P_0 = X$$

$$\begin{aligned} P_{k+1} &= P_k \cap \text{proj}_x \{ (x, u) \mid Ax + Bu \in P_k \} \\ &= P_k \cap \text{proj}_x [A, B]^{-1} P_k \end{aligned}$$

Not scalable...

$[A, B]^{-1}$	easy with $\mathcal{H}$ -representation
proj_x	easy with $\mathcal{V}$ -representation
$\cap$	easy with $\mathcal{H}$ -representation

Unconstrained control to algebraic system

Linear Discrete **Control** System (DT**CS**)

$$x_{k+1} = Ax_k + Bu_k, \quad x_k \in X$$

Linear Discrete **Algebraic** System (DT**AS**)

$$Ex_{k+1} = Ax_k, \quad x_k \in X$$

Invariant Set $S \subseteq X$ if

$$\forall x \in S, \exists y \in S : Ey = Ax$$

Reduction

The set S is controlled invariant for the DT**CS** iff it is invariant for the DT**AS**

$$\Pi x_{k+1} = \Pi Ax_k, \quad x_k \in X$$

where Π is a projection on $\text{Im}(B)^\perp$.

Ellipsoidal controlled invariant Set of Algebraic Systems

$$Ex_{k+1} = Ax_k, \quad x_k \in X$$

Convex set S invariant iff

$$AS \subseteq ES$$

$$(AS)^\circ \supseteq (ES)^\circ$$

$$A^{-T}S^\circ \supseteq E^{-T}S^\circ$$

Ellipsoidal computation

$$C^\circ = \{x \mid x^T Q x \leq 1\} \quad \Rightarrow \quad A^{-T}C^\circ = \{x \mid x^T AQA^T x \leq 1\}$$

S-procedure:

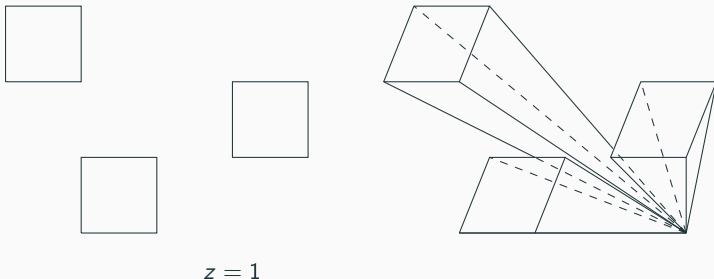
$$EQE^T - AQA^T \text{ is PSD.}$$

Generalization to Hybrid Systems

Main obstacle : need to homogenize

Safe set in each modes can have different centers (The issue is already present for non-Hybrid System if the safe set is not symmetric).

Homogenization with new variable $z = 1$.



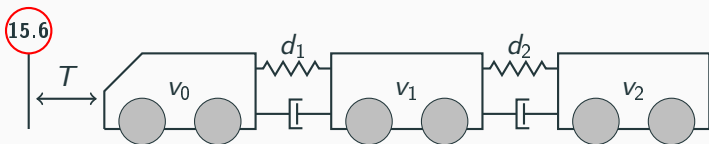
We now look for **cones** K instead of **set** S .

We replace **polar** S° with **dual** K^* .

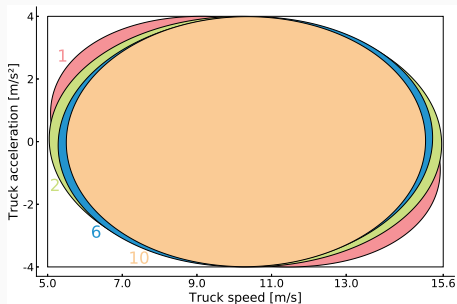
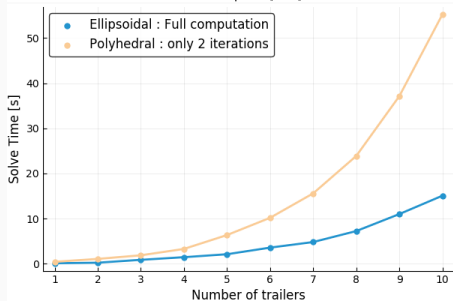
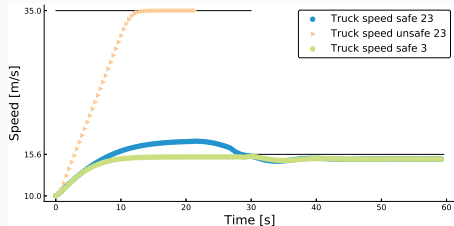
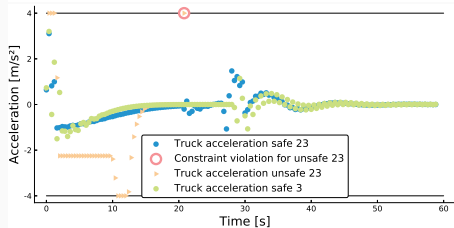
Back to our example

Four speed limits with a delay T to satisfy limit

→ Hybrid System



Results



Conclusion

Conclusion

- Safe MPC either needs long horizon or precomputed controlled invariant sets;
- Polyhedral computation of controlled invariant sets is computationally intensive;
- Ellipsoidal computation of controlled invariant sets is semidefinite representable in the dual space.

Future work

- Semialgebraic invariant sets using Sum-of-Squares.
- Improve BMI optimization for non-symmetric case.
- Piecewise quadratic invariant set.