

Certifying unstability of Switched Systems using Sum of Squares Programming

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Motivating application

Simulation error $\iff$ Dynamical system¹

$$\dot{x} = Ax$$

Euler method:

$$x_{k+1} = x_k + hAx_k$$

Midpoint method:

$$x_{k+1} = x_k + hA\left(x_k + \frac{h}{2}Ax_k\right)$$



Euler method:

$$e_{k+1} = (I + hA)e_k + e_{\text{local}}$$

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¹Based on joint work with Cláudio Gomes from the University of Antwerp

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What happens if we **switch** between the methods or step sizes ?
Can we detect “bad” switching sequences ?

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Motivating example

Consider

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

We have

$$A \cdot A \cdot A \cdots = 0,$$

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(in fact, A and B are **nilpotent**, so $A^2 = B^2 = 0$), but...

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The asymptotic **growth rate** of this sequence is 2, i.e.

$$\lim_{k \rightarrow \infty} \|(A \cdot B)^{\frac{k}{2}}\|^{\frac{1}{k}} = 2$$

Joint Spectral Radius and Switched Systems

Switched Systems

$$x_{k+1} = A_{\sigma_k} x_k, \quad \sigma_k \in \{1, \dots, m\}.$$

$$\text{Switched System stable} \iff \lim_{k \rightarrow \infty} x_k = 0 \quad \forall x_0.$$

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Joint Spectral Radius (JSR) $\mathcal{A} = \{A_1, \dots, A_m\}$

$$\rho(\mathcal{A}) := \limsup_{k \rightarrow \infty} \max_{\sigma \in \{1, \dots, m\}} \|A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}\|^{1/k}.$$

$$\text{Switched System stable} \iff \rho(\mathcal{A}) < 1$$

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Switched System **stable** $\iff \rho(\mathcal{A}) < 1$

- If $m = 1$, “standard” spectral radius $\rho(A_1) = \max_i |\lambda_i|$.
- If $m \geq 2$, much more complicated...
 - Determining if $\rho(\mathcal{A}) < 1$ is **NP-hard** (Tsitsiklis & Blondel 1997).
 - Determining if $\rho(\mathcal{A}) \leq 1$ is **undecidable** (Blondel & Tsitsiklis 2000).

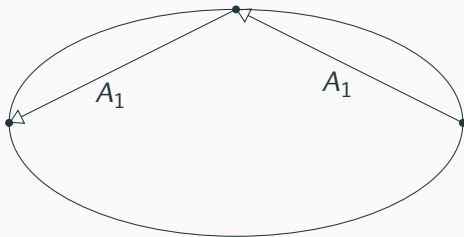
Joint Spectral Radius and Invariant Sets

Invariant Set

$$A_{\sigma}x \in S, \quad \forall x \in S, \sigma \in \{1, \dots, m\}$$

Example

$$m = 1, A_1 = \text{Rotation}(90^\circ) \times \begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix}$$



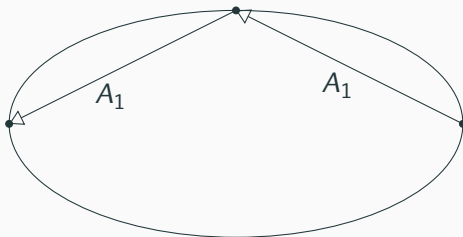
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$$\exists \text{ invariant set for } A_1/\gamma, \dots, A_m/\gamma \quad \implies \quad \rho(\mathcal{A}) \leq \gamma$$

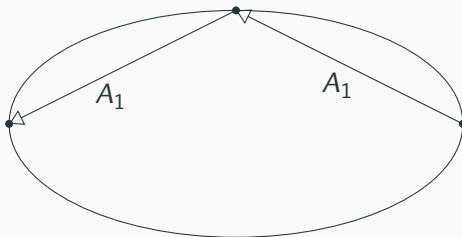
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SOS- $2d$ invariant sets

Can compute invariant sets using Sum of Squares Programming (SOS) : $S = \{x : p(x) \leq 1\}$ where $p(x)$ homogeneous SOS degree $2d$.

P.A. Parrilo and A. Jadbabaie, Approximation of the joint spectral radius using sum of squares, *Linear Algebra and its Applications*, Vol. 428, No. 10, pp. 2385–2402, 2008.

SOS- $2d$ invariant sets

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Guarantee

$$\rho_{\text{SOS-}2d}(\mathcal{A}) = \inf_{\gamma} \gamma$$

s.t. $\exists$ SOS- $2d$ invariant set for $A_1/\gamma, \dots, A_m/\gamma$

$$\eta^{-\frac{1}{2d}} \rho_{\text{SOS-}2d}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \rho_{\text{SOS-}2d}(\mathcal{A})$$

where $\eta = \min \left\{ m, \binom{n+d-1}{d} \right\}$ and $A_\sigma \in \mathbb{R}^{n \times n}$.

As $d \rightarrow \infty$, $\eta^{-\frac{1}{2d}} \rightarrow 0$.

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Measures

Measures μ defined on the sphere

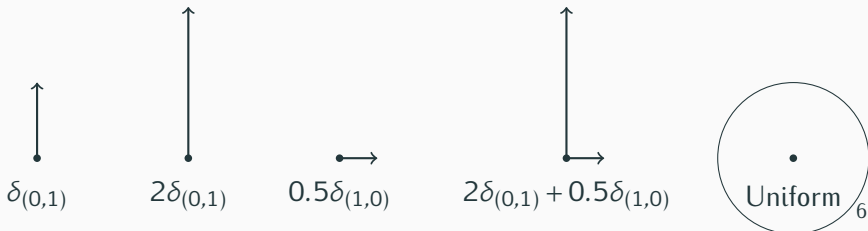
$$\mathbb{S}^{n-1} = \{x : \|x\|_2 = 1\}$$

For $B \in \mathbb{S}^{n-1}$, $\mu(B)$ is **measure** of B .

atomic $\Leftrightarrow$ sum of **Diracs**

amplitude in drawing represents

- **mass** of atom for *atomic* measure,
- **density** otherwise.



Transfer/Frobenius-Perron operator and push-forward measure

Usual definition:

$$(A\#\mu)(B) = \mu(A^{-1}(B))$$

Problem: $\mathbb{S}^{n-1}$ not invariant under A_σ .

Solution: Alternative definition:

$$\int_{\mathbb{S}^{n-1}} f(x)(A\#\mu)(dx) = \int_{\mathbb{S}^{n-1}} f(Ax)\mu(dx) \quad \forall f.$$

Example

$$A_1 = \text{Rotation}(90^\circ) \times \begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix}$$



Periodic trajectory

Periodic trajectory of growth rate γ :

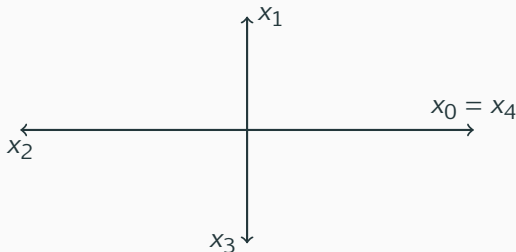
$$\gamma^k x_0 = A^k x_0$$

$\exists x_i, 1 \leq i \leq k$, such that:

$$\gamma x_{i+1} = Ax_i$$

Example

$$A = \text{Rotation}(90^\circ) \times \begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix}, x_0 = (1, 0), k = 4$$



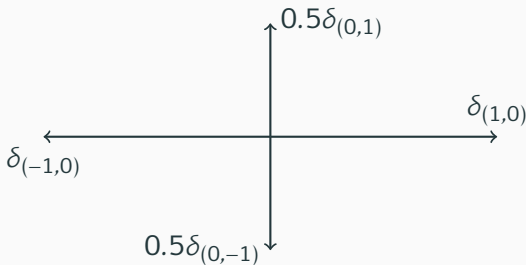
Periodic trajectory as atomic measure I

Equivalent to:

$$\gamma\delta_{x_{k+1}} = A\#\delta_{x_k}$$

Example

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Periodic trajectory as atomic measure II

Convexification: Find μ_k such that for all $k = 1, \dots$

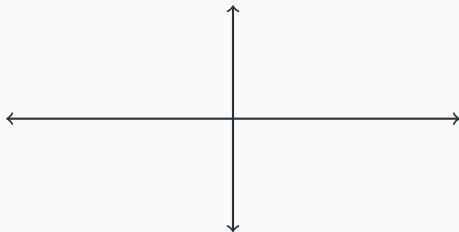
$$\gamma\mu_{k+1} = A\#\mu_k.$$

Project the time:

$$\gamma\mu = A\#\mu.$$

Example

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$$\mu = A\#\mu$$

Periodic trajectory as atomic measure III

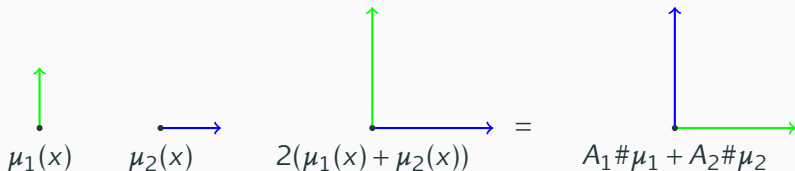
$$\gamma(\mu_1 + \mu_2) = A_1 \# \mu_1 + A_2 \# \mu_2.$$

periodic trajectory of growth rate $\gamma \leftrightarrow$ atomic measures.

Example

$$A_1 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$



Joint Spectral Radius and Invariant Measures

Theorem

Let μ_σ , $\sigma = 1, \dots, m$, be measures on $\mathbb{S}^{n-1}$, that satisfies

$$\sum_{\sigma=1}^m A_\sigma \# \mu_\sigma \geq \gamma \sum_{\sigma=1}^m \mu_\sigma,$$

and at least one of the expectations is nonzero.

Then $\rho(\mathcal{A}) \geq \gamma$.

SOS- $2d$ invariant measures

Moments $\tilde{\mu}$ of degree $2d$. Can write constraint

$$\langle \tilde{\mu}, p(x) \rangle = \int_{\mathbb{S}^{n-1}} p(x) d\tilde{\mu} \geq 0, \quad \forall p(x) \text{ SOS of degree } 2d.$$

as Semidefinite Programming constraint.

Dual to invariant sets program.

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- Can **efficiently** detect when moments of **atomic** measures.
- **relaxation**: can have moments of no measure ! But...

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pseudo-measure

$$\langle \tilde{\mu}, p(x) \rangle \geq 0, \quad \forall p(x) \text{ SOS.}$$

Making a liar tell the truth

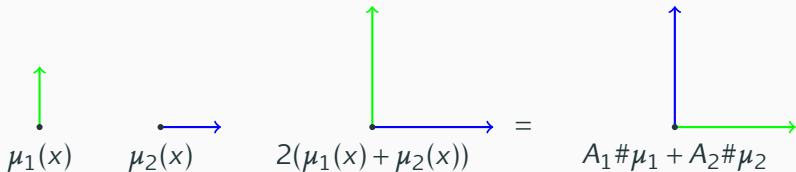
If **atomic**, take x_0 :

$$\sigma_{k+1} = \arg \max \mu_{\sigma}(\{x_k / \|x_k\|_2\}), \quad x_{k+1} = A_{\sigma_{k+1}} x_k$$

Example

$$A_1 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$



If **pseudo-measure**, take $p_0(x)$ SOS:

$$\sigma_{k+1} = \underset{\sigma}{\operatorname{argmax}} \langle \mu_{\sigma}, p_k(x) \rangle, \quad p_{k+1}(x) = p_k(A_{\sigma_{k+1}} x)$$

Note: Switching built backward!

Infinite backward sequence

Guarantee

$$m^{-\frac{1}{2d}} \rho_{\text{SOS-}2d}(\mathcal{A}) \leq \lim_{k \rightarrow \infty} \|A_{\sigma_1} \cdots A_{\sigma_k}\|^{\frac{1}{k}} \leq \rho(\mathcal{A}) \leq \rho_{\text{SOS-}2d}(\mathcal{A}).$$

If sequence **periodic** $\rightarrow$ can compute growth rate

B. L., R.M. Jungers and P.A. Parrilo, Generating unstable trajectories for Switched Systems via Dual Sum-Of-Squares techniques, *Proceedings of the 19th International Conference on Hybrid Systems: Computation and Control*, ACM, pp. 51–60, 2016.

Constraint Joint Spectral Radius

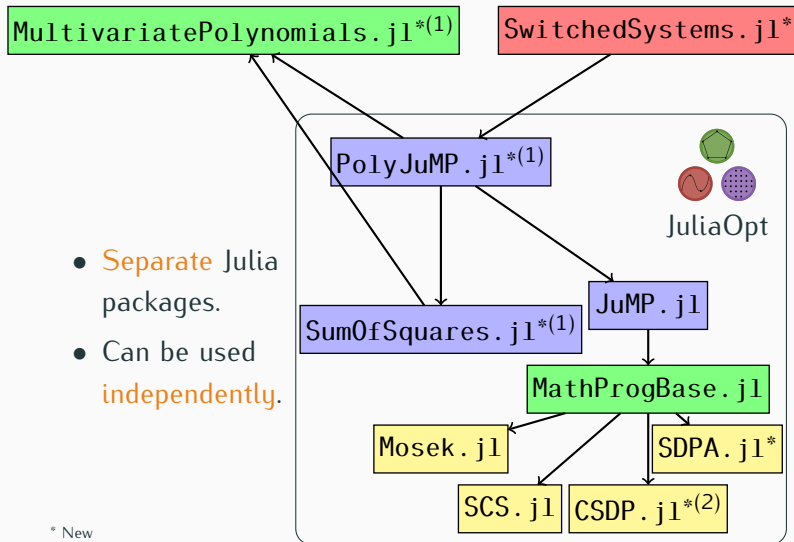
Allowed switching sequences $\sigma_1, \sigma_2, \dots \leftrightarrow$ paths of graph/automaton.

$$\rho(A)^{-\frac{1}{2d}} \rho_{\text{SOS-}2d}(G, \mathcal{A}) \leq \lim_{k \rightarrow \infty} \|A_{\sigma_1} \cdots A_{\sigma_k}\|^{\frac{1}{k}} \leq \rho(G, \mathcal{A}) \leq \rho_{\text{SOS-}2d}(G, \mathcal{A}).$$

where A is adjacency matrix.

Implementation

- **Separate** Julia packages.
- Can be used **independently**.



^{*} New

(1) Joint work with Robin Deits, Joey Huchette, Chris Coey and Amelia Perry of MIT

(2) Joint work with Elias Kuthe of Universität Duisburg Essen

Results

- Lower bound **certificate** using measures.
- Detection of **atomic** measures and extraction of **periodic** trajectory.
- Algorithm for producing **guaranteed** high-growth sequences.
- Guarantee depending on **topology** of the automaton.
- **Convex** program provides both upper and lower bounds.

Open questions

- Improved **guarantees**? What is best possible?
- Is the sequence always **periodic** for a fixed d ? Can we bound the period ?

Why is it a relaxation ?

The Semidefinite Programming constraints on the truncated moments of μ_σ are necessary for the existence of a measure with those moments, and **any** value of the moments of higher degree (not necessarily 0).

In the Common Quadratic Lyapunov Function case, i.e. $2d = 2$, those conditions are necessary and sufficient.

- For every truncated moments, we can find a μ_σ that has these moments.
- And there exists a slack variable μ_{slack} such that

$$\sum_{\sigma=1}^m \mathbb{E}_{\mu_\sigma}[p(A_\sigma x)] - \underline{\gamma}^{2d} \sum_{\sigma=1}^m \mathbb{E}_{\mu_\sigma}[p(x)] = \mathbb{E}_{\mu_{\text{slack}}}, \forall p(x) \in \Sigma_{2d},$$

so

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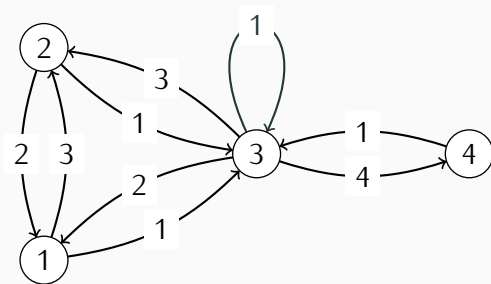
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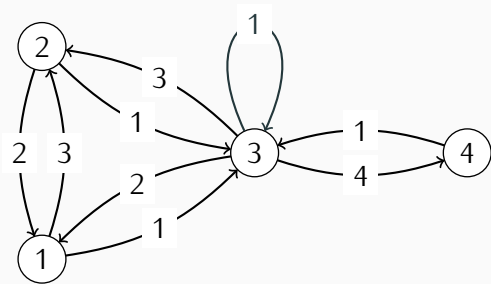
$$\sum_{\sigma=1}^m \mathcal{A}_\sigma \mu_\sigma - \underline{\gamma}^{2d} \sum_{\sigma=1}^m \mu_\sigma \text{ has the same truncated moments than } \mu_{\text{slack}}$$

Adjacency matrix



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

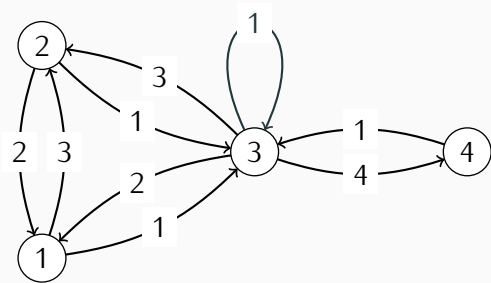
Adjacency matrix



$$\rho(A) = \lim_{k \rightarrow \infty} \|A^k\|^{1/k}.$$

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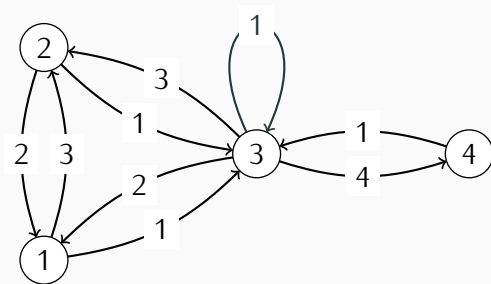


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$$\|A\|_{\infty} = \max \begin{pmatrix} 0+1+1+0 \\ 1+0+1+0 \\ 1+1+1+1 \\ 0+0+1+0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 1 \end{pmatrix}.$$

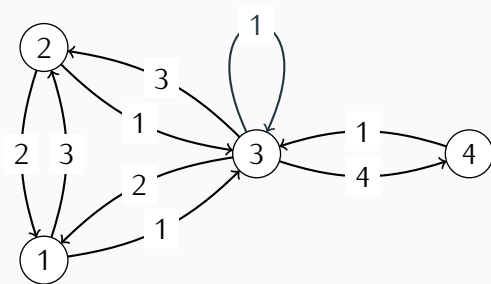
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$$A^2 = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 2 & 4 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \|A^2\|_{\infty} = \max \begin{pmatrix} 2+1+2+1 \\ 1+2+2+1 \\ 2+2+4+1 \\ 1+1+1+1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 9 \\ 4 \end{pmatrix}.$$

Adjacency matrix



$$\rho(A) = \lim_{k \rightarrow \infty} \|A^k\|^{1/k}.$$

$$A^3 = \begin{pmatrix} 3 & 4 & 6 & 2 \\ 4 & 3 & 6 & 2 \\ 6 & 6 & 9 & 4 \\ 2 & 2 & 4 & 1 \end{pmatrix}.$$

$$\|A^3\|_{\infty} = \max \begin{pmatrix} 3+4+6+2 \\ 4+3+6+2 \\ 6+6+9+4 \\ 2+2+4+1 \end{pmatrix} = \begin{pmatrix} 15 \\ 15 \\ 25 \\ 9 \end{pmatrix}.$$