

Generating unstable trajectories for Switched Systems via Dual Sum-Of-Squares techniques

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Introduction

$$x_{k+1} = A_{\sigma_k} x_k$$

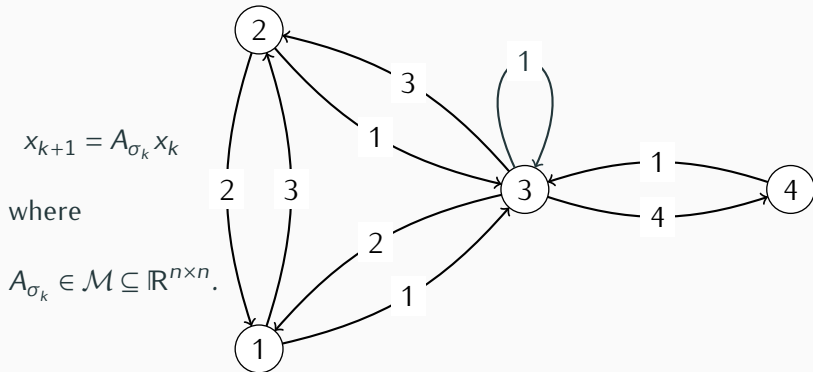
where

$$A_{\sigma_k} \in \mathcal{M} \subseteq \mathbb{R}^{n \times n}.$$

Joint Spectral Radius

$$\rho = \lim_{k \rightarrow \infty} \max \|A_{\sigma_k} \cdots A_{\sigma_1}\|^{1/k}.$$

Constrained Switched Systems

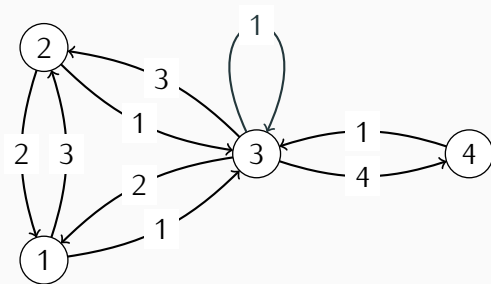


Constrained Joint Spectral Radius

$$\rho = \lim_{k \rightarrow \infty} \max \|A_{\sigma_k} \cdots A_{\sigma_1}\|^{1/k}.$$

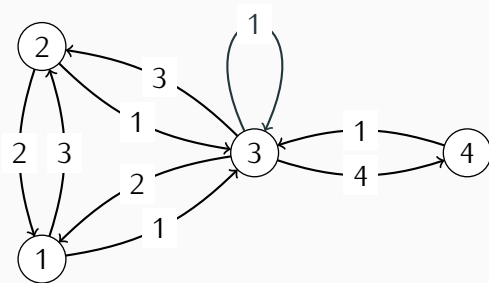
$\sigma_1, \dots, \sigma_k$ is a path.

Adjacency matrix



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

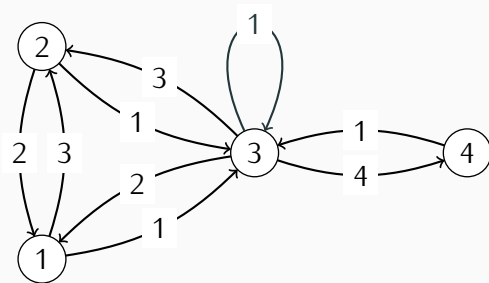
Adjacency matrix



$$\rho(A) = \lim_{k \rightarrow \infty} \|A^k\|^{1/k}.$$

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Adjacency matrix

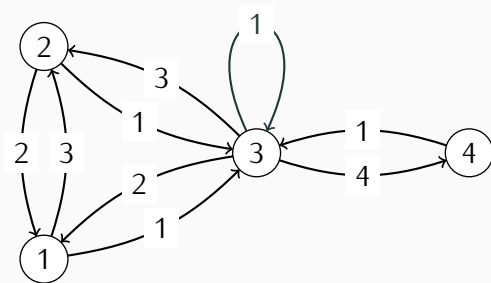


$$\rho(A) = \lim_{k \rightarrow \infty} \|A^k\|^{1/k}.$$

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$\|A\|_{\infty} = \max \begin{pmatrix} 0+1+1+0 \\ 1+0+1+0 \\ 1+1+1+1 \\ 0+0+1+0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 1 \end{pmatrix}.$$

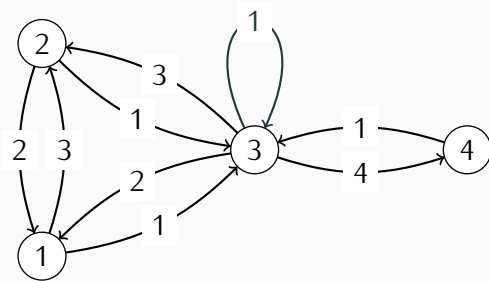
Adjacency matrix



$$\rho(A) = \lim_{k \rightarrow \infty} \|A^k\|^{1/k}.$$

$$A^2 = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 2 & 4 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \|A^2\|_{\infty} = \max \begin{pmatrix} 2+1+2+1 \\ 1+2+2+1 \\ 2+2+4+1 \\ 1+1+1+1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 9 \\ 4 \end{pmatrix}.$$

Adjacency matrix



$$\rho(A) = \lim_{k \rightarrow \infty} \|A^k\|^{1/k}.$$

$$A^3 = \begin{pmatrix} 3 & 4 & 6 & 2 \\ 4 & 3 & 6 & 2 \\ 6 & 6 & 9 & 4 \\ 2 & 2 & 4 & 1 \end{pmatrix}.$$

$$\|A^3\|_{\infty} = \max \begin{pmatrix} 3+4+6+2 \\ 4+3+6+2 \\ 6+6+9+4 \\ 2+2+4+1 \end{pmatrix} = \begin{pmatrix} 15 \\ 15 \\ 25 \\ 9 \end{pmatrix}.$$

Current guarantee

$$\binom{n+d-1}{d}^{-\frac{1}{2d}} \rho_{\text{SOS},2d} \leq \rho \leq \rho_{\text{SOS},2d}.$$

New guarantee

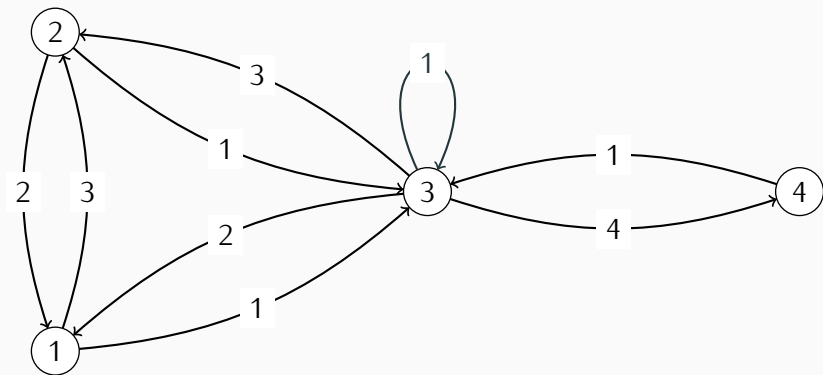
$$\rho(\mathbf{A})^{-\frac{1}{2d}} \rho_{\text{SOS},2d} \leq \rho \leq \rho_{\text{SOS},2d}.$$

New algorithm generating a sequence $\sigma_1, \dots, \sigma_k$ such that

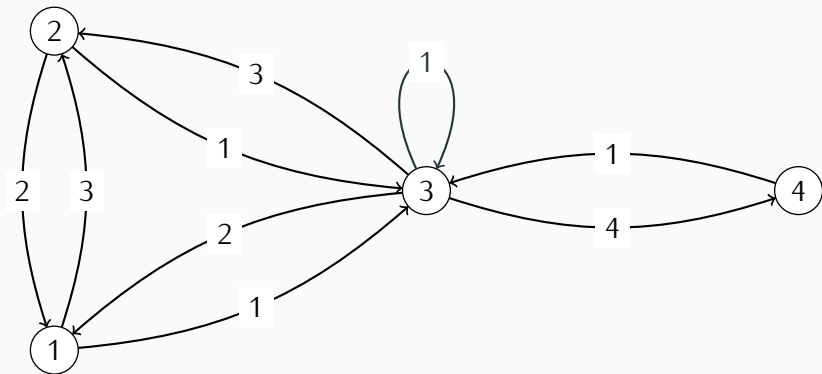
$$\rho(\mathbf{A})^{-\frac{1}{2d}} \rho_{\text{SOS},2d} \leq \lim_{k \rightarrow \infty} \|A_{\sigma_1} \cdots A_{\sigma_k}\|^{\frac{1}{k}} \leq \rho.$$

Computing upper bounds using Sum of Squares Programming

Upper bounds for ρ



Upper bounds for ρ



$$u \rightarrow_{\sigma} v \quad \|A_{\sigma}x\|_v \leq \gamma \|x\|_u.$$

Is it enough ?

Is it enough ? YES

[Philippe et al. 2015]

For any $\epsilon > 0$, there exists a set of norms $\{\|\cdot\|_v : v \in V\}$ such that

$$\|A_\sigma x\|_v \leq (\rho + \epsilon) \|x\|_u$$

holds for all edge $(u, v, \sigma) \in E$.

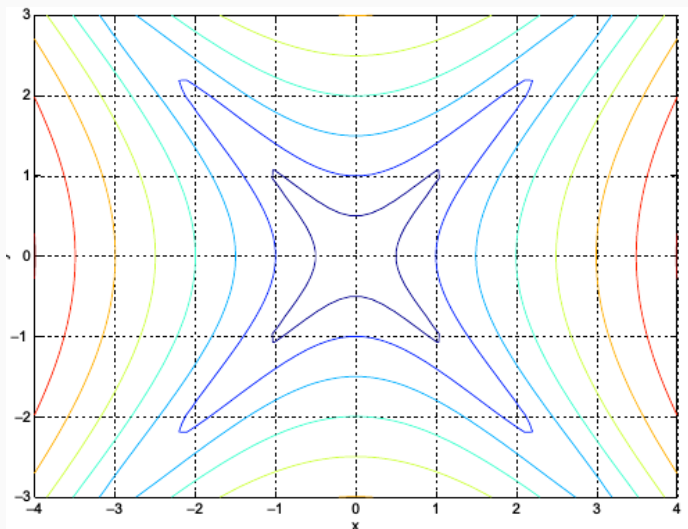
Why norms ?

- Homogeneous,
- Positive,
- Convex

Why norms ?

- Homogeneous,
- Positive,
- ~~Convex~~ not needed

Homogeneous positive polynomial



$$(x_1^2 - x_2^2)^2 + \varepsilon(x_1^2 + x_2^2)^2$$

Sum of Squares relaxation [Parrilo and Jadbabaie 2008]

Barvinok's theorem

$\forall \|\cdot\|, \exists p(x)$ SOS degree $2d$ such that

$$p(x)^{\frac{1}{2d}} \leq \|x\| \leq \binom{n+d-1}{d}^{\frac{1}{2d}} p(x)^{\frac{1}{2d}}.$$

$$\|A_\sigma x\|_v \leq \gamma \|x\|_u \rightarrow p_v(A_\sigma x) \leq \gamma^{2d} \binom{n+d-1}{d} p_u(x)$$

Sum of Squares relaxation [Parrilo and Jadbabaie 2008]

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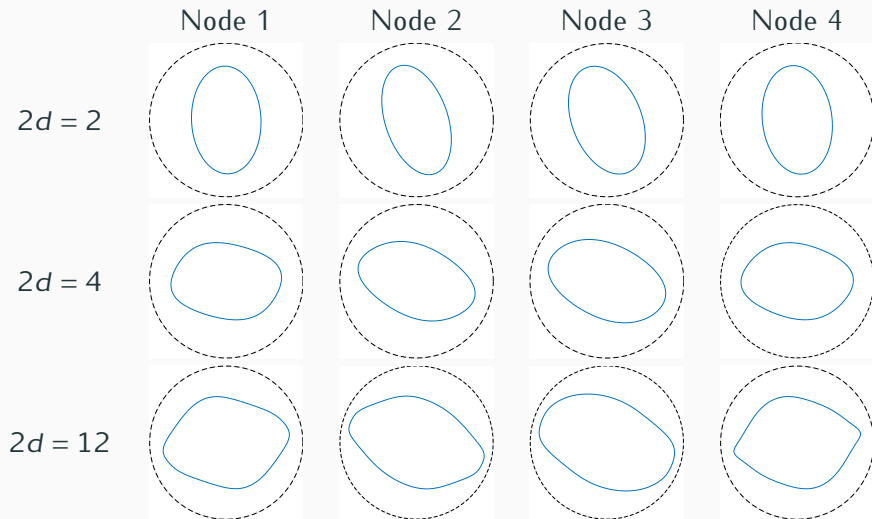
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More tricky

$$\gamma^{2d} \binom{n+d-1}{d} p_u(x) - p_v(A_\sigma x) \text{ is SOS ?}$$

Lyapunov functions



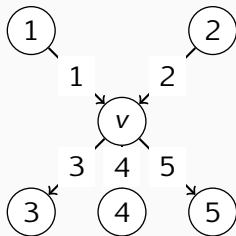
$$\begin{array}{ll} \inf \gamma & \\ p_v(x) \text{ is SOS} & \forall v \in V \\ \gamma^{2d} p_u(x) - p_v(A_\sigma x) \text{ is SOS} & \forall (u, v, \sigma) \in E. \end{array}$$

$$\langle \widetilde{\mathbb{E}}, p(x) \rangle = \widetilde{\mathbb{E}}[p(x)] = \int_{\mathbb{S}^{n-1}} p(x) d\widetilde{\mu} \geq 0.$$

$\forall p(x)$ SOS.

Dual constraint: More flow in than out

$$\sum_{(u,v,\sigma) \in E} \widetilde{\mathbb{E}}_{uv\sigma}[p(A_\sigma x)] \geq \gamma^{2d} \sum_{(v,w,\sigma) \in E} \widetilde{\mathbb{E}}_{vw\sigma}[p(x)], \quad \forall v \in V, p(x) \text{ SOS},$$

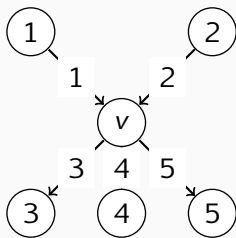


$$\widetilde{\mathbb{E}}_{1v1}[p(A_1 x)] + \widetilde{\mathbb{E}}_{2v2}[p(A_2 x)] \geq \gamma^{2d} (\widetilde{\mathbb{E}}_{v33}[p(x)] + \widetilde{\mathbb{E}}_{v44}[p(x)] + \widetilde{\mathbb{E}}_{v55}[p(x)]),$$

Generating high growth sequences

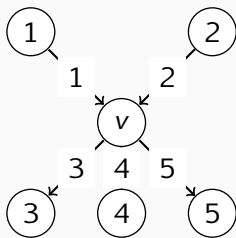
Building a high growth sequence

$$\sum_{(u,v,\sigma) \in E} \widetilde{\mathbb{E}}_{uv\sigma}[p(A_\sigma x)] \geq \gamma^{2d} \sum_{(v,w,\sigma) \in E} \widetilde{\mathbb{E}}_{vw\sigma}[p(x)], \quad \forall v \in V, p(x) \text{ SOS},$$



Building a high growth sequence

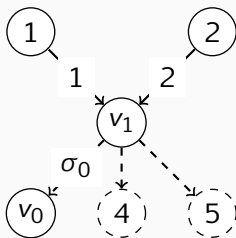
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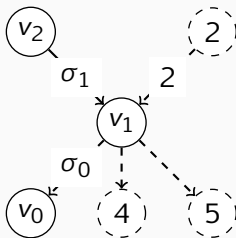
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$$\begin{aligned} \widetilde{\mathbb{E}}_{1v_1}[p(A_1x)] + \widetilde{\mathbb{E}}_{2v_2}[p(A_2x)] &\geq \gamma^{2d} (\widetilde{\mathbb{E}}_{v_33}[p(x)] + \widetilde{\mathbb{E}}_{v_44}[p(x)] + \widetilde{\mathbb{E}}_{v_55}[p(x)]), \\ \widetilde{\mathbb{E}}_{1v_1}[p(A_1x)] + \widetilde{\mathbb{E}}_{2v_2}[p(A_2x)] &\geq \gamma^{2d} \widetilde{\mathbb{E}}_{v_1v_0\sigma_0}[p(x)], \end{aligned}$$

Building a high growth sequence

$$\sum_{(u,v,\sigma) \in E} \widetilde{\mathbb{E}}_{uv\sigma}[p(A_\sigma x)] \geq \gamma^{2d} \sum_{(v,w,\sigma) \in E} \widetilde{\mathbb{E}}_{vw\sigma}[p(x)], \quad \forall v \in V, p(x) \text{ SOS},$$

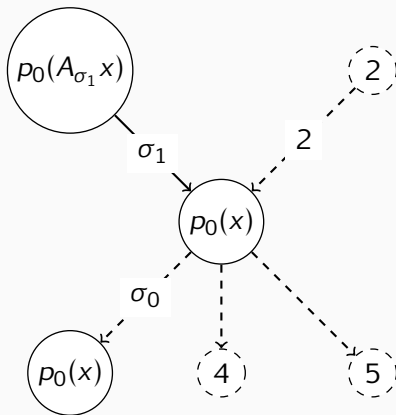


$$\begin{aligned} \widetilde{\mathbb{E}}_{1v_1}[p(A_1x)] + \widetilde{\mathbb{E}}_{2v_2}[p(A_2x)] &\geq \gamma^{2d} (\widetilde{\mathbb{E}}_{v_33}[p(x)] + \widetilde{\mathbb{E}}_{v_44}[p(x)] + \widetilde{\mathbb{E}}_{v_55}[p(x)]), \\ \widetilde{\mathbb{E}}_{1v_1}[p(A_1x)] + \widetilde{\mathbb{E}}_{2v_2}[p(A_2x)] &\geq \gamma^{2d} \widetilde{\mathbb{E}}_{v_1v_0\sigma_0}[p(x)], \\ \widetilde{\mathbb{E}}_{v_2v_1\sigma_1}[p(A_1x)] &\geq \frac{1}{2} \gamma^{2d} \widetilde{\mathbb{E}}_{v_1v_0\sigma_0}[p(x)]. \end{aligned}$$

Choice of polynomial

$p_0(x)$ random.

$$p_{k+1}(x) = p_k(A_{\sigma_k} x) = p_0(A_{\sigma_1} \cdots A_{\sigma_k} x)$$



Guarantee on sequence produced

Let

$$\theta_k \triangleq \widetilde{\mathbb{E}}_{v_{k+1} v_k \sigma_k} [p_0(A_{\sigma_1} \cdots A_{\sigma_k} x)].$$

We have

$$\theta_k \geq \frac{\gamma^{2d}}{d^{-(v_k)}} \theta_{k-1}.$$

Guarantee on sequence produced

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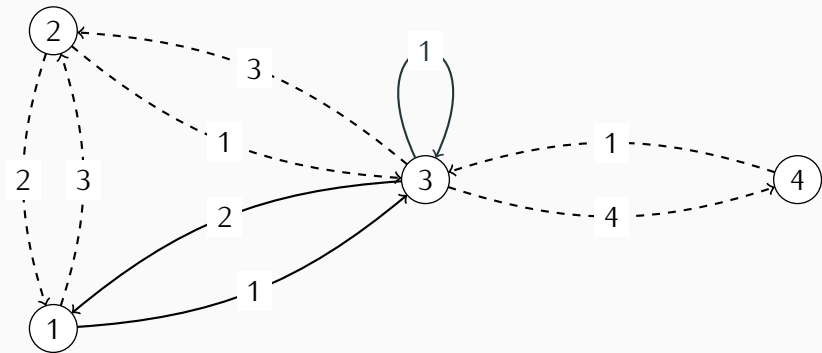
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We have

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$$\frac{\gamma}{\|\mathbf{A}^\top\|_\infty^{\frac{1}{2d}}} \leq \lim_{k \rightarrow \infty} \|A_{\sigma_1} \cdots A_{\sigma_k}\|_\infty^{\frac{1}{k}}.$$

Let's try it

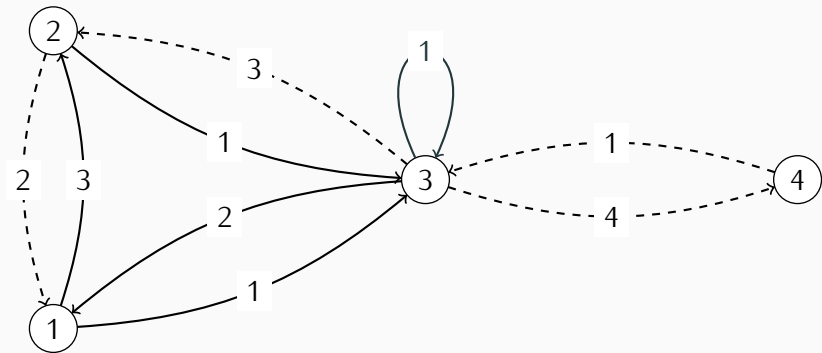


$$2d = 2$$

$3 \rightarrow_2 1 \rightarrow_1 3 \rightarrow_2 1 \rightarrow_3 1 \rightarrow_1 3 \rightarrow_1 3 \rightarrow_1 3 \rightarrow_1 3$

0.97289

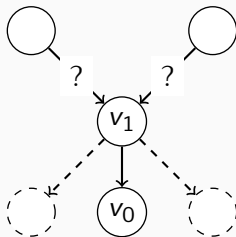
Let's try it



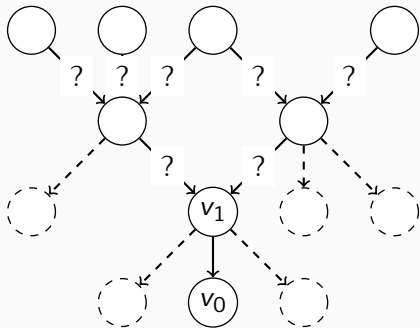
$$2d = 12$$

$$3 \rightarrow_2 1 \rightarrow_1 3 \rightarrow_2 1 \rightarrow_3 2 \rightarrow_1 3 \rightarrow_1 3 \rightarrow_1 3 \rightarrow_1 3$$

$$0.97482$$



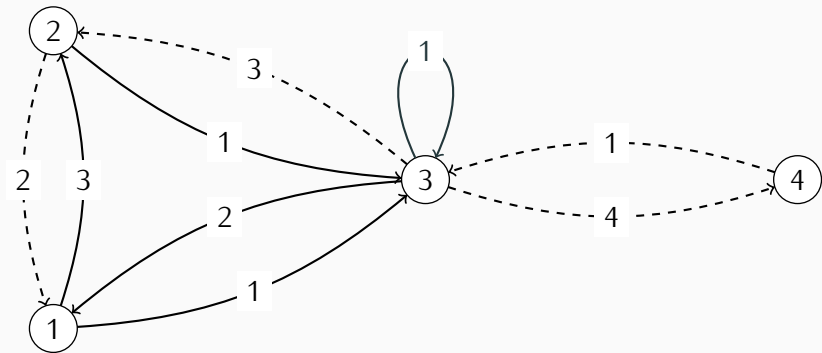
Improvement



With horizon l :

$$\frac{\gamma}{\|[\mathbf{A}^\top]^l\|_\infty^{\frac{1}{2dl}}} \leq \lim_{k \rightarrow \infty} \|A_{\sigma_1} \cdots A_{\sigma_k}\|_2^{\frac{1}{k}}.$$

Let's try it



$$2d = 2, l = 3$$

$$3 \rightarrow_2 1 \rightarrow_1 3 \rightarrow_2 1 \rightarrow_3 2 \rightarrow_1 3 \rightarrow_1 3 \rightarrow_1 3 \rightarrow_1 3$$

$$0.97482$$

Guarantee for the upper bound

As $l \rightarrow \infty$,

$$\rho(\mathbf{A})^{-\frac{1}{2d}} \rho_{\text{SOS},2d} \leq \lim_{k \rightarrow \infty} \|A_{\sigma_1} \cdots A_{\sigma_k}\|^{\frac{1}{k}} \leq \rho \leq \rho_{\text{SOS},2d}.$$

Improving the lower bound:

- Use $\mathbf{A}(G')$ where edges of 0 dual variable are removed.
- Find lower bound in an high growth rate sequence produced.

Conclusion

Results

- Algorithm producing sequence of growth rate asymptotically close to ρ .
- Guarantee on the “closeness” depending on d and $\rho(\mathbf{A})$.
- Provides improved lower bounds.

Open questions

- Can we improve guarantee, e.g. using n ?
- Is the sequence always periodic for a fixed d ? Can we bound the period ?