

Generating unstable trajectories for Switched Systems via Dual Sum-Of-Squares techniques

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Introduction

Lyapunov functions and their Sum of Squares Approximation

Measures containing unstable trajectories and their moment relaxation

Conclusion

Introduction

Motivation (from a Simulation viewpoint)

Analysis of integration method for $\dot{x} = Ax$ equivalent to analysis of dynamical systems¹:

Euler method:

$$x_{k+1} = x_k + hAx_k$$

Midpoint method:

$$x_{k+1} = x_k + hA\left(x_k + \frac{h}{2}Ax_k\right)$$



Euler method:

$$e_{k+1} = (I + hA)e_k + e_{\text{local}}$$

Midpoint method:

$$e_{k+1} = \left(I + hA + \frac{h^2}{2}A^2\right)e_k + e_{\text{local}}$$

¹Based on joint work with Cláudio Gomes from the University of Antwerp

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What happens if we **switch** between the methods or step sizes ?
How can we compute the set of allowed switching sequences ?

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Example

Consider

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

We have

$$\lim_{k \rightarrow \infty} A \cdot A \cdot A \cdots = 0,$$

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$$\lim_{k \rightarrow \infty} \|A \cdot B \cdot A \cdot B \cdots\| = \infty \qquad (!)$$

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$$\lim_{k \rightarrow \infty} \|A \cdot B \cdot A \cdot B \cdots\| = \infty \qquad (!)$$

The asymptotic growth rate of this sequence is 2, i.e.

$$\lim_{k \rightarrow \infty} \|(A \cdot B)^{\frac{k}{2}}\|^{\frac{1}{k}} = 2$$

Joint spectral radius

Given a set of $n \times n$ matrices $\Sigma = \{A_1, \dots, A_m\}$, what is the maximum “growth rate” that can be achieved by **arbitrary switching**?

$$\rho(\Sigma) := \limsup_{k \rightarrow \infty} \max_{\sigma \in \{1, \dots, m\}} \|A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}\|^{1/k}.$$

Appeared in several different contexts: linear algebra (Rota–Strang 1960), wavelets (Daubechies–Lagarias 1992), switched linear systems, etc.

- If $m = 1$, “standard” spectral radius $\rho(A_1) = \max_i |\lambda_i|$.
- If $m \geq 2$, much more complicated...

Joint spectral radius is hard

Determining if $\rho(\Sigma) \leq 1$ is NP-hard (Tsitsiklis & Blondel 1997).

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Related work by Gurvitz (1995), Kozyakin, Barabanov, Wang-Lagarias “finiteness conjecture”, Blondel-Nesterov, Theys, Jungers, etc...

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Still, how to compute/approximate $\rho(\Sigma)$?

What approximation guarantees can we have?

How to produce “bad” switching sequences?

$$x_{k+1} = A_{\sigma_k} x_k$$

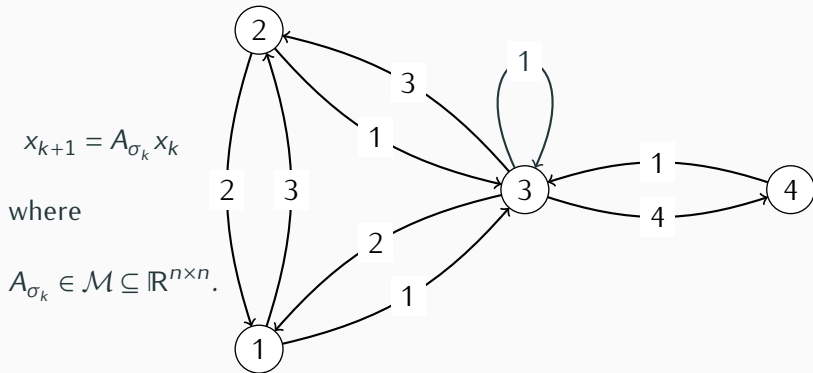
where

$$A_{\sigma_k} \in \mathcal{M} \subseteq \mathbb{R}^{n \times n}.$$

Joint Spectral Radius

$$\rho(\mathcal{A}) = \lim_{k \rightarrow \infty} \max \|A_{\sigma_k} \cdots A_{\sigma_1}\|^{1/k}.$$

Constrained Switched Systems

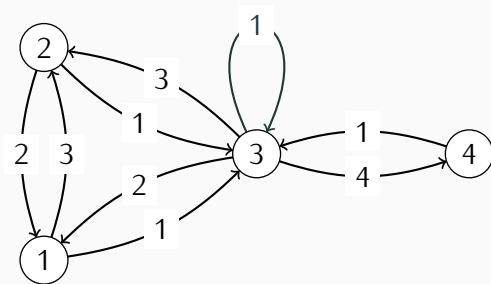


Constrained Joint Spectral Radius

$$\rho(G, \mathcal{A}) = \lim_{k \rightarrow \infty} \max \|A_{\sigma_k} \cdots A_{\sigma_1}\|^{1/k}.$$

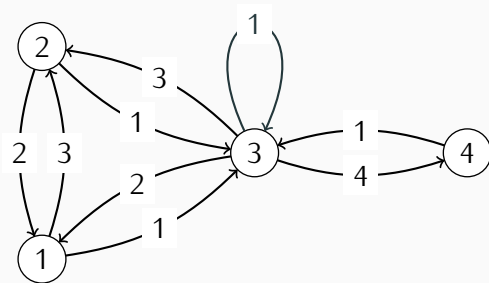
$\sigma_1, \dots, \sigma_k$ is a path.

Adjacency matrix



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

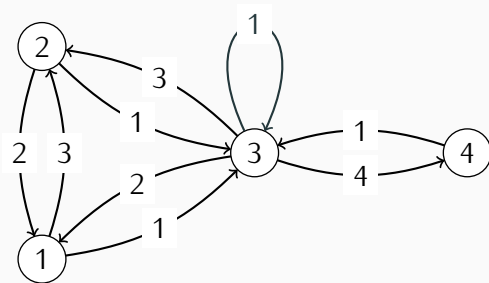
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$$\rho(A) = \lim_{k \rightarrow \infty} \|A^k\|^{1/k}.$$

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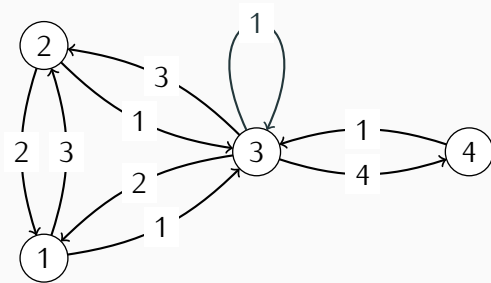


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$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$\|A\|_{\infty} = \max \begin{pmatrix} 0+1+1+0 \\ 1+0+1+0 \\ 1+1+1+1 \\ 0+0+1+0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 1 \end{pmatrix}.$$

Adjacency matrix

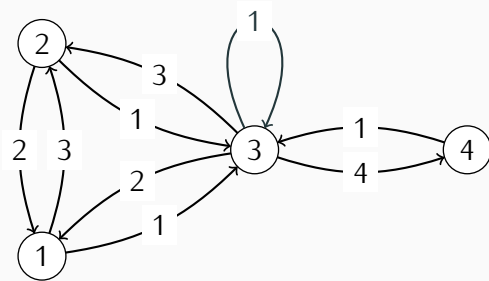


$$\rho(A) = \lim_{k \rightarrow \infty} \|A^k\|^{1/k}.$$

$$A^2 = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 2 & 4 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

$$\|A^2\|_{\infty} = \max \begin{pmatrix} 2+1+2+1 \\ 1+2+2+1 \\ 2+2+4+1 \\ 1+1+1+1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 9 \\ 4 \end{pmatrix}.$$

Adjacency matrix



$$\rho(A) = \lim_{k \rightarrow \infty} \|A^k\|^{1/k}.$$

$$A^3 = \begin{pmatrix} 3 & 4 & 6 & 2 \\ 4 & 3 & 6 & 2 \\ 6 & 6 & 9 & 4 \\ 2 & 2 & 4 & 1 \end{pmatrix}.$$

$$\|A^3\|_{\infty} = \max \begin{pmatrix} 3+4+6+2 \\ 4+3+6+2 \\ 6+6+9+4 \\ 2+2+4+1 \end{pmatrix} = \begin{pmatrix} 15 \\ 15 \\ 25 \\ 9 \end{pmatrix}.$$

Lyapunov functions and their Sum of Squares Approximation

Theorem

Let $f(x)$ be a strictly positive homogeneous function, that satisfies

$$f(A_{\sigma}x) \leq \bar{\gamma}f(x), \quad \forall x \in \mathbb{R}^n, \sigma = 1, \dots, m.$$

Then $\rho(\mathcal{A}) \leq \bar{\gamma}$.

We need a homogeneous polynomials $p(x)$ that satisfies

$$p(x) \geq 0, \quad \bar{\gamma}^{2d} p(x) - p(A_\sigma x) \geq 0 \quad \sigma = 1, \dots, m.$$

While this is not tractable, we can use instead

$$p(x) \text{ is SOS}, \quad \bar{\gamma}^{2d} p(x) - p(A_\sigma x) \text{ is SOS} \quad \sigma = 1, \dots, m.$$

Then $\rho(\mathcal{A}) \leq \rho_{\text{SOS-2d}}(\mathcal{A}) := \bar{\gamma}$.

For fixed $\bar{\gamma}$, this is an SOS program, can be solved using an SDP solver.

Example

Based on (Ando and Shih 98). Consider the two matrices:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},$$

- The true spectral radius is $\rho(A_1, A_2) = 1$.
- A common quadratic Lyapunov function (i.e. $d = 2$), gives $\rho(A_1, A_2) \leq \sqrt{2}$.

Example (cont.)

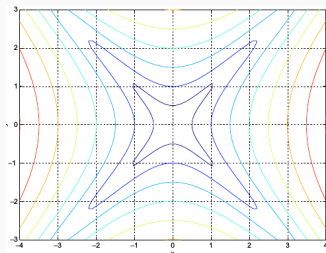
A quartic SOS Lyapunov function is enough to prove an upper bound of $1 + \epsilon$ for *every* $\epsilon > 0$, since

$$V(x) = (x_1^2 - x_2^2)^2 + \epsilon(x_1^2 + x_2^2)^2$$

satisfies

$$(1 + \epsilon)V(x) - V(A_1x) = (x_2^2 - x_1^2 + \epsilon(x_1^2 + x_2^2))^2$$

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Example (cont.)

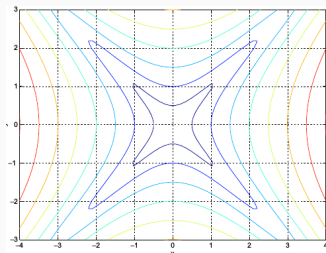
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What (if anything) can we say about approximation properties?

SOS Guarantees

The degree- d SOS upper bound for $\rho(A_1, \dots, A_m)$ satisfies

$$\eta^{-\frac{1}{2d}} \rho_{\text{SOS-2d}}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \rho_{\text{SOS-2d}}(\mathcal{A})$$

where $\eta = \min \left\{ m, \binom{n+d-1}{d} \right\}$.

- As degree $d \rightarrow \infty$, approx. ratio goes to 1!
- For unbounded m , related to Barvinok/John's ellipsoid.
- For fixed n , proof based on Lyapunov iteration.

P.A. Parrilo and A. Jadbabaie, Approximation of the joint spectral radius using sum of squares, *Linear Algebra and its Applications*, Vol. 428, No. 10, pp. 2385–2402, 2008.

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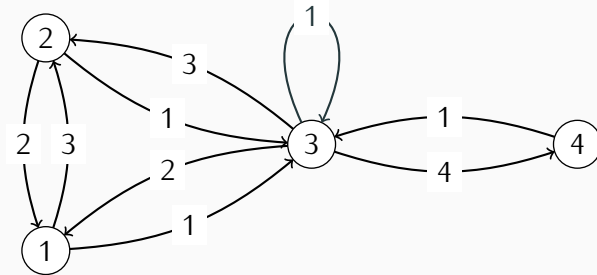
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But...

- How to generalize to *constrained* switching?
- How to produce “bad” switching sequences?

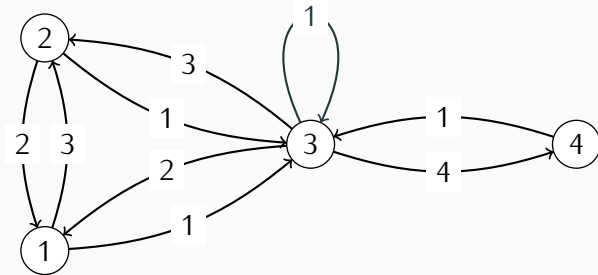
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Constrained Switching Systems and multi-Lyapunov



M. Philippe, R. Essick, G. Dullerud and R.M. Jungers, Stability of discrete-time switching systems with constrained switching sequences, *arXiv preprint arXiv:1503.06984*, 2015

Constrained Switching Systems and multi-Lyapunov



$$u \rightarrow_{\sigma} v \quad f_v(A_{\sigma}x) \leq \bar{\gamma} f_u(x).$$

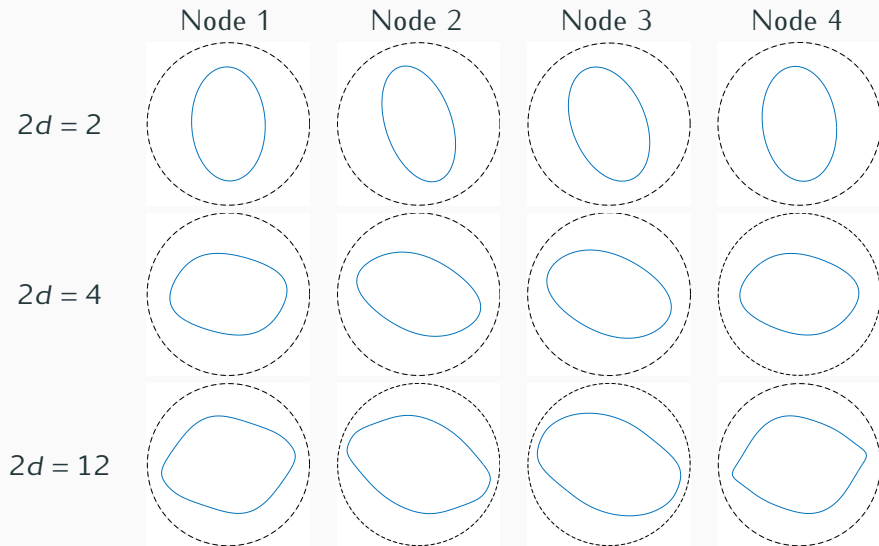
The degree- d SOS upper bound satisfies

$$\eta^{-\frac{1}{2d}} \rho_{\text{SOS-2d}}(G, \mathcal{A}) \leq \rho(G, \mathcal{A}) \leq \rho_{\text{SOS-2d}}(G, \mathcal{A})$$

where $\eta = \binom{n+d-1}{d}$.

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multi-Lyapunov functions



$$\begin{array}{ll} \inf \bar{\gamma} & \\ p_v(x) \text{ is SOS} & \forall v \in V \\ \bar{\gamma}^{2d} p_u(x) - p_v(A_\sigma x) \text{ is SOS} & \forall (u, v, \sigma) \in E. \end{array}$$

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Last but not least...

- How to produce “bad” switching sequences?

Measures containing unstable trajectories and their moment relaxation

Measures containing unstable trajectories

Theorem

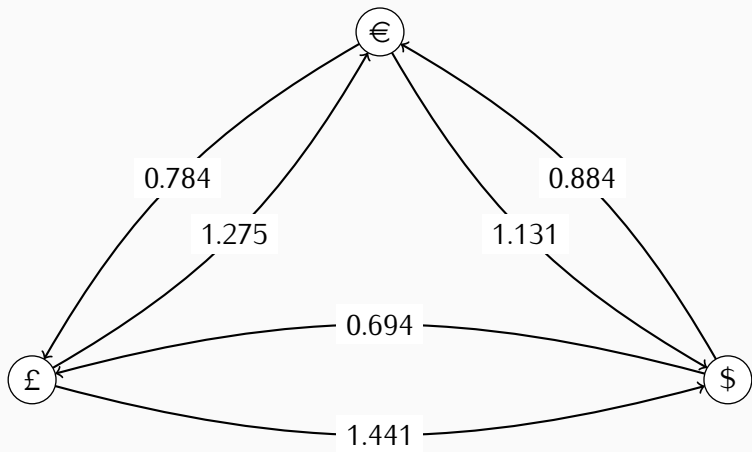
Let μ_σ , $\sigma = 1, \dots, m$, be measures on $\mathbb{S}^{n-1}$, that satisfies

$$\sum_{\sigma=1}^m \mathbb{E}_{\mu_\sigma}[f(A_\sigma x)] \geq \underline{\gamma} \sum_{\sigma=1}^m \mathbb{E}_{\mu_\sigma}[f(x)], \quad \forall \text{ nonnegative homogeneous } f$$

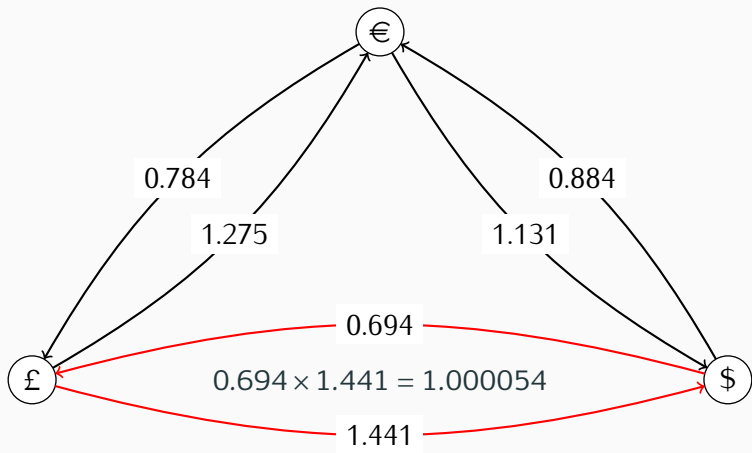
and at least one of the expectations is nonzero.

Then $\rho(\mathcal{A}) \geq \underline{\gamma}$.

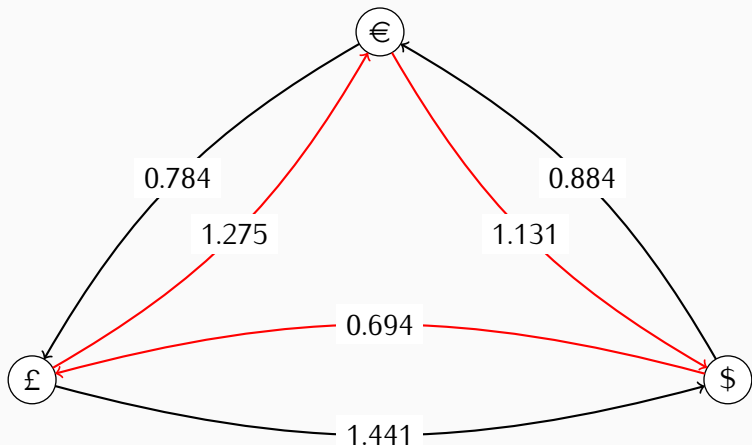
Example: Arbitration in Economy



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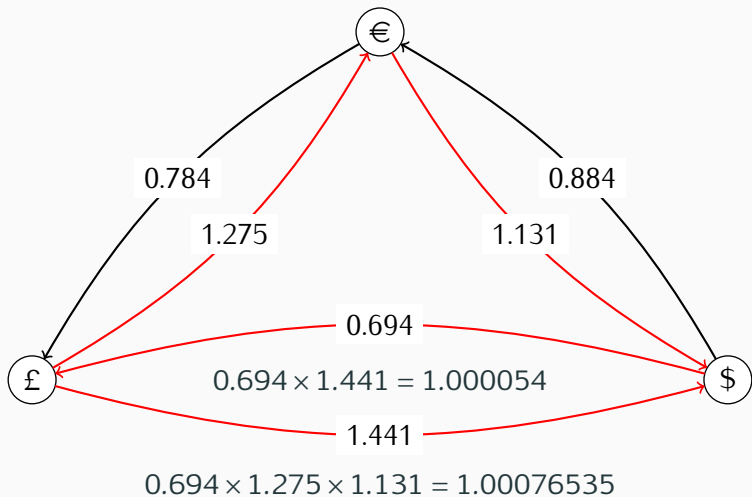


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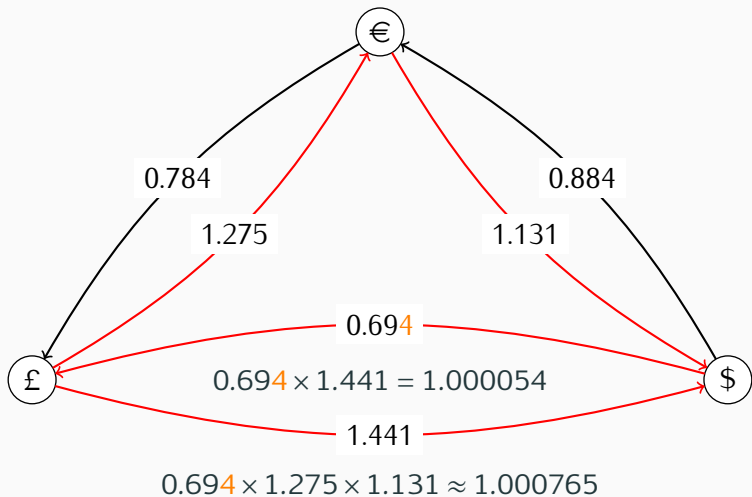


$$0.694 \times 1.275 \times 1.131 \approx 1.000765$$

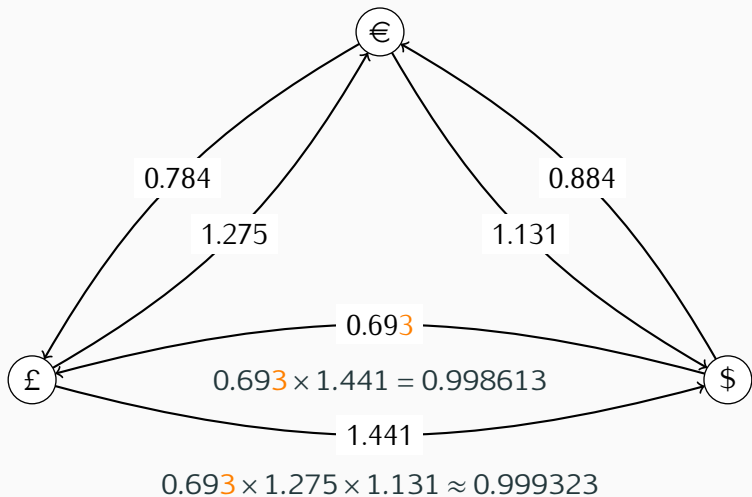
Let's make it stable



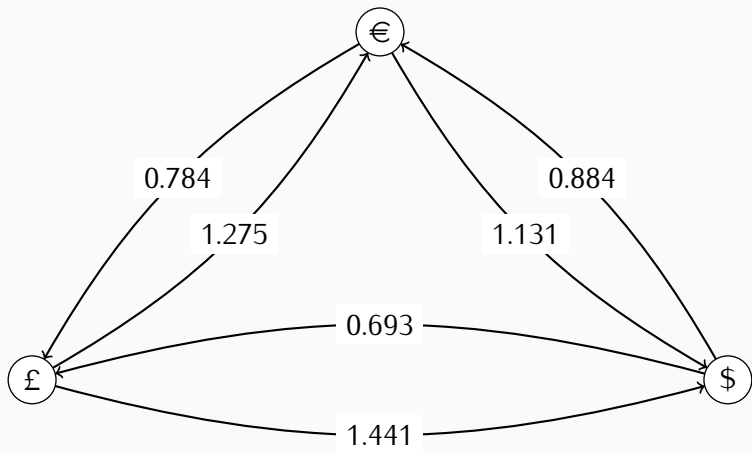
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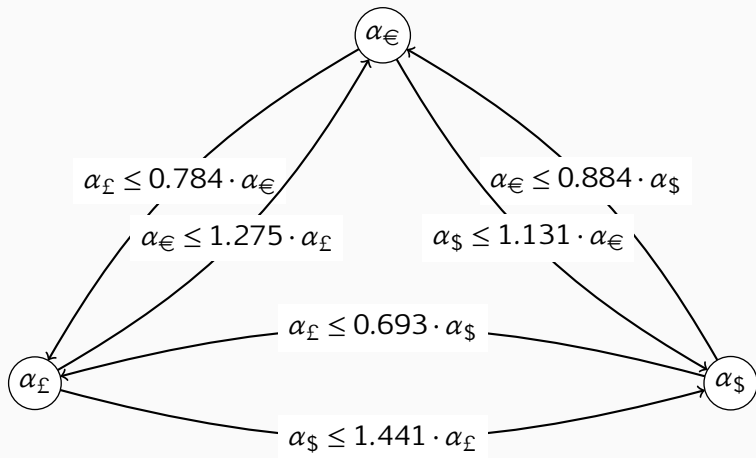
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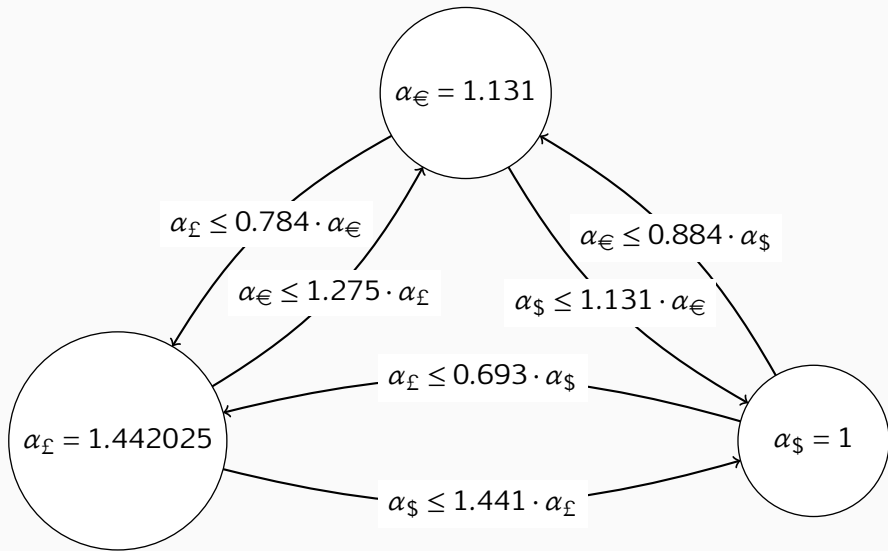
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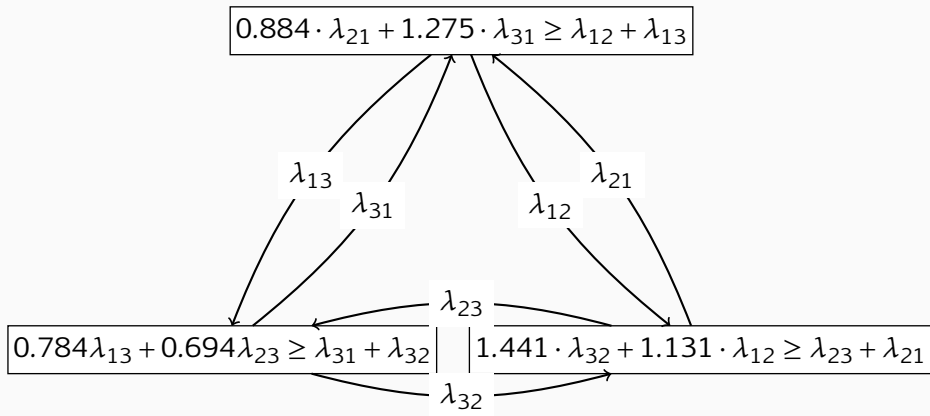
Lyapunov functions in the scalar case



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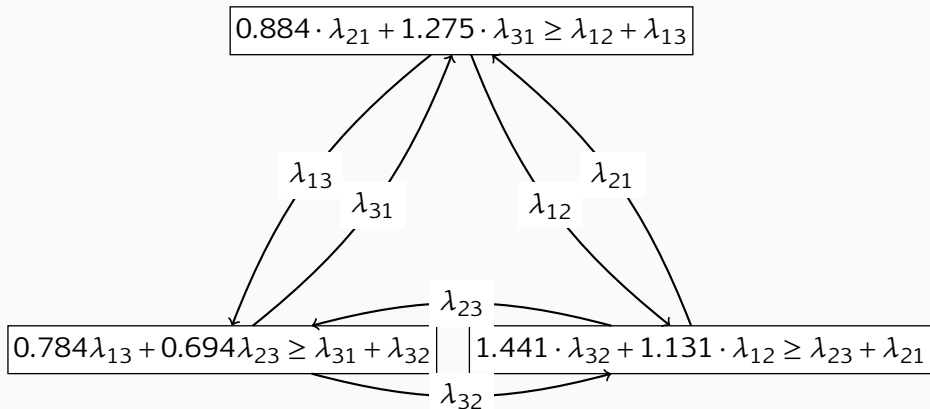


Measures containing arbitrages (LP dual of Lyapunov program)



Solutions are:

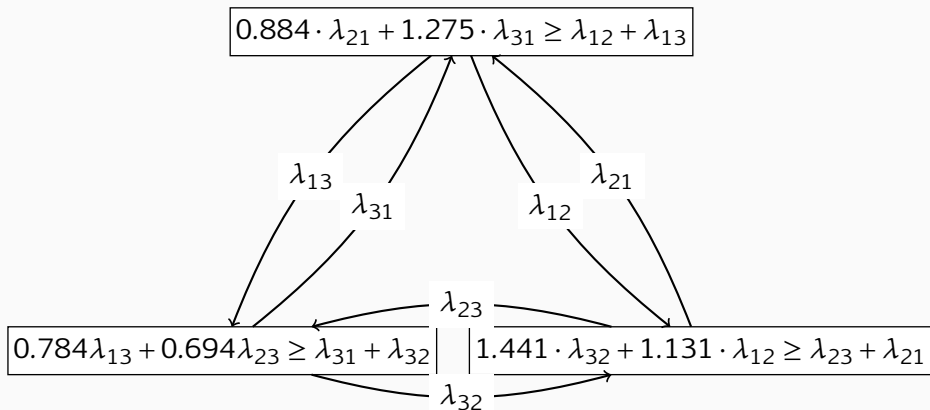
Measures containing arbitrages (LP dual of Lyapunov program)



Solutions are: Any conic combinations of

$$(\lambda_{23}, \lambda_{32}) = (1, 0.694) \text{ and } (\lambda_{23}, \lambda_{31}, \lambda_{12}) = (1, 0.694, 0.694 \times 1.274)$$

Measures containing arbitrages (LP dual of Lyapunov program)



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The measures can contain **several** different trajectories!

Extracting trajectories from measures

We say that the measures contain only one trajectory if for every $\hat{x}$, at *most* one of the terms of the LHS of

$$\sum_{\sigma=1}^m \mathbb{E}_{\mu_{\sigma}}[f(A_{\sigma}x)] \geq \underline{\gamma} \sum_{\sigma=1}^m \mathbb{E}_{\mu_{\sigma}}[f(x)],$$

is nonzero with $f(x) = \delta_{\hat{x}}(x)$.

If the measures contain only one trajectory, we can extract it as follows

- Take σ_0, x_0 such that $\mathbb{E}_{\mu_{\sigma_0}}[\delta_{x_0}] > 0$.
- Set σ_{k+1} to the **only** σ such that $\mathbb{E}_{\mu_{\sigma}}[\delta_{x_k}(A_{\sigma}x)] > 0$.
- Set $x_{k+1} = A_{\sigma_{k+1}}^{-1} x_k$.

Remark: the trajectory is built in **backward** order.

Moment Program

Given the moments of a measure μ up to degree $2d$, we can write the constraint

$$\mathbb{E}_{\mu}[p(x)] = \int_{\mathbb{S}^{n-1}} p(x) d\mu \geq 0, \quad \forall p(x) \text{ SOS of degree } 2d.$$

as Semidefinite Programming constraints [Lasserre 2001].

The moment relaxation of our program is:

$$\begin{aligned} & \sup_{\mu_{uv\sigma} \in \Sigma_{2d}^*, \underline{\gamma} \in \mathbb{R}} \underline{\gamma} \\ & \sum_{(u,v,\sigma) \in E} \mathbb{E}_{\mu_{uv\sigma}}[p(A_{\sigma}x)] \geq \underline{\gamma}^{2d} \sum_{(v,w,\sigma) \in E} \mathbb{E}_{\mu_{vw\sigma}}[p(x)], \forall p(x) \in \Sigma_{2d}, \forall v \in V, \\ & \sum_{(u,v,\sigma) \in E} \mathbb{E}_{\mu_{uv\sigma}} \left[\sum_{i=1}^n x_i^{2d} \right] = 1. \end{aligned}$$

Why is it a relaxation ?

The Semidefinite Programming constraints on the truncated moments of μ_σ are necessary for the existence of a measure with those moments, and **any** value of the moments of higher degree (not necessarily 0).

In the Common Quadratic Lyapunov Function case, i.e. $2d = 2$, those conditions are necessary and sufficient.

- For every truncated moments, we can find a μ_σ that has these moments.
- And there exists a slack variable μ_{slack} such that

$$\sum_{\sigma=1}^m \mathbb{E}_{\mu_\sigma}[p(A_\sigma x)] - \underline{\gamma}^{2d} \sum_{\sigma=1}^m \mathbb{E}_{\mu_\sigma}[p(x)] = \mathbb{E}_{\mu_{\text{slack}}}, \forall p(x) \in \Sigma_{2d},$$

so

$$\sum_{\sigma=1}^m \mathcal{A}_\sigma \mu_\sigma - \underline{\gamma}^{2d} \sum_{\sigma=1}^m \mu_\sigma = \mu_{\text{slack}}$$

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so

$$\sum_{\sigma=1}^m \mathcal{A}_\sigma \mu_\sigma - \underline{\gamma}^{2d} \sum_{\sigma=1}^m \mu_\sigma \text{ has the same truncated moments than } \mu_{\text{slack}}$$

When we have luck

If we find measures such that

$$\sum_{\sigma=1}^m \mathcal{A}_\sigma \mu_\sigma - \underline{\gamma}^{2d} \sum_{\sigma=1}^m \mu_\sigma$$

is a measure then $\underline{\gamma} \leq \rho(\mathcal{A})$.

- In fact, the moment and lyapunov programs are dual, so $\underline{\gamma} = \overline{\gamma}$. Therefore, we have $\underline{\gamma} = \rho(\mathcal{A}) = \overline{\gamma}$.
- In practice we can try to decompose the μ_σ 's as a sum of diracs. But it can only works if the system has the finiteness property.

Example

Let

$$\mathcal{A} = \{A_1 = e_1 e_2^\top, A_2 = e_2 e_1^\top\}$$

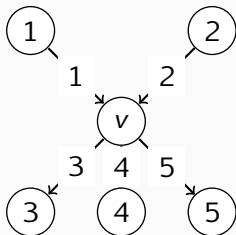
A Lyapunov with $2d = 2$ and $\bar{\gamma} = 1$ is given by $\|x\|_2$.

Measures with $2d = 2$ and $\underline{\gamma} = 1$ are given by $\mu_1 = \delta_{(0,1)}$ and $\mu_2 = \delta_{(1,0)}$.

$$\mu_1(x) + \mu_2(x) \geq \mu_1(A_1^{-1}x) + \mu_2(A_2^{-1}x)$$

Building a high growth sequence

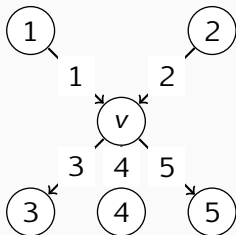
$$\sum_{(u,v,\sigma) \in E} \mathbb{E}_{\mu_{uv\sigma}}[p(A_\sigma x)] \geq \underline{\gamma}^{2d} \sum_{(v,w,\sigma) \in E} \mathbb{E}_{\mu_{vw\sigma}}[p(x)], \quad \forall v \in V, p(x) \text{ SOS},$$



Construct sequence *backwards*, using “best expectation”:

Building a high growth sequence

$$\sum_{(u,v,\sigma) \in E} \mathbb{E}_{\mu_{uv\sigma}}[p(A_\sigma x)] \geq \underline{\gamma}^{2d} \sum_{(v,w,\sigma) \in E} \mathbb{E}_{\mu_{vw\sigma}}[p(x)], \quad \forall v \in V, p(x) \text{ SOS},$$

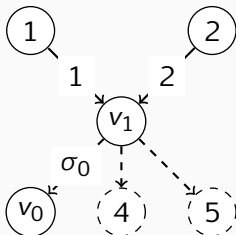


Construct sequence *backwards*, using “best expectation”:

$$\mathbb{E}_{\mu_{1v1}}[p(A_1 x)] + \mathbb{E}_{\mu_{2v2}}[p(A_2 x)] \geq \underline{\gamma}^{2d} (\mathbb{E}_{\mu_{v33}}[p(x)] + \mathbb{E}_{\mu_{v44}}[p(x)] + \mathbb{E}_{\mu_{v55}}[p(x)]),$$

Building a high growth sequence

$$\sum_{(u,v,\sigma) \in E} \mathbb{E}_{\mu_{uv\sigma}}[p(A_\sigma x)] \geq \underline{\gamma}^{2d} \sum_{(v,w,\sigma) \in E} \mathbb{E}_{\mu_{vw\sigma}}[p(x)], \quad \forall v \in V, p(x) \text{ SOS},$$

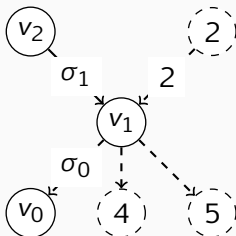


Construct sequence *backwards*, using “best expectation”:

$$\begin{aligned} \mathbb{E}_{\mu_{1v_1}}[p(A_1 x)] + \mathbb{E}_{\mu_{2v_2}}[p(A_2 x)] &\geq \underline{\gamma}^{2d} (\mathbb{E}_{\mu_{v_33}}[p(x)] + \mathbb{E}_{\mu_{v_44}}[p(x)] + \mathbb{E}_{\mu_{v_55}}[p(x)]), \\ \mathbb{E}_{\mu_{1v_1}}[p(A_1 x)] + \mathbb{E}_{\mu_{2v_12}}[p(A_2 x)] &\geq \underline{\gamma}^{2d} \mathbb{E}_{\mu_{v_1v_0\sigma_0}}[p(x)], \end{aligned}$$

Building a high growth sequence

$$\sum_{(u,v,\sigma) \in E} \mathbb{E}_{\mu_{uv\sigma}}[p(A_\sigma x)] \geq \underline{\gamma}^{2d} \sum_{(v,w,\sigma) \in E} \mathbb{E}_{\mu_{vw\sigma}}[p(x)], \quad \forall v \in V, p(x) \text{ SOS},$$



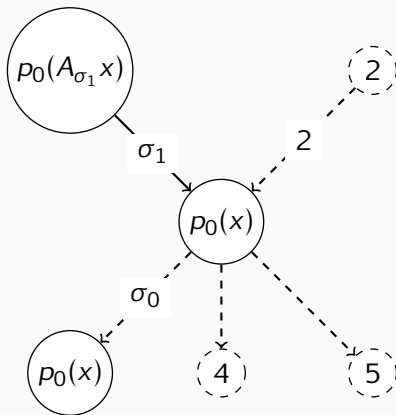
Construct sequence *backwards*, using “best expectation”:

$$\begin{aligned} \mathbb{E}_{\mu_{1v_1}}[p(A_1 x)] + \mathbb{E}_{\mu_{2v_2}}[p(A_2 x)] &\geq \underline{\gamma}^{2d} (\mathbb{E}_{\mu_{v_33}}[p(x)] + \mathbb{E}_{\mu_{v_44}}[p(x)] + \mathbb{E}_{\mu_{v_55}}[p(x)]), \\ \mathbb{E}_{\mu_{1v_1}}[p(A_1 x)] + \mathbb{E}_{\mu_{2v_2}}[p(A_2 x)] &\geq \underline{\gamma}^{2d} \mathbb{E}_{\mu_{v_1 v_0 \sigma_0}}[p(x)], \\ \mathbb{E}_{\mu_{v_2 v_1 \sigma_1}}[p(A_1 x)] &\geq \frac{1}{2} \underline{\gamma}^{2d} \mathbb{E}_{\mu_{v_1 v_0 \sigma_0}}[p(x)]. \end{aligned}$$

Choice of polynomial

Initial $p_0(x)$ arbitrary (as long as it is strictly SOS).

$$p_{k+1}(x) = p_k(A_{\sigma_k} x) = p_0(A_{\sigma_1} \cdots A_{\sigma_k} x)$$



Algorithm and Guarantees

Pick an arbitrary polynomial $p_0(x) \in \text{int}(\Sigma_{2d})$

Pick an edge $(v_1, v_0, \sigma_0) \in E$ such that $\mathbb{E}_{\mu_{v_1 v_0 \sigma_0}}[p_0(x)] > 0$

Set $p_1(x) \leftarrow p_0(x)$

for $k = 1, 2, \dots$ **do**

$(v_{k+1}, v_k, \sigma_k) \leftarrow \arg \max_{(u, v_k, \sigma) \in E} \mathbb{E}_{\mu_{uv_k \sigma}}[p_k(A_\sigma x)]$

$p_{k+1} \leftarrow p_k(A_{\sigma_k} x)$

end for

Let

$$\theta_k \triangleq \mathbb{E}_{\mu_{v_{k+1} v_k \sigma_k}}[p_0(A_{\sigma_1} \cdots A_{\sigma_k} x)].$$

We have

$$\theta_k \geq \frac{\gamma^{2d}}{d^{-(v_k)}} \theta_{k-1}.$$

Algorithm and Guarantees

Pick an arbitrary polynomial $p_0(x) \in \text{int}(\Sigma_{2d})$

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Set $p_1(x) \leftarrow p_0(x)$

for $k = 1, 2, \dots$ **do**

$(v_{k+1}, v_k, \sigma_k) \leftarrow \arg \max_{(u, v_k, \sigma) \in E} \mathbb{E}_{\mu_{u v_k \sigma}}[p_k(A_\sigma x)]$

$p_{k+1} \leftarrow p_k(A_{\sigma_k} x)$

end for

Let

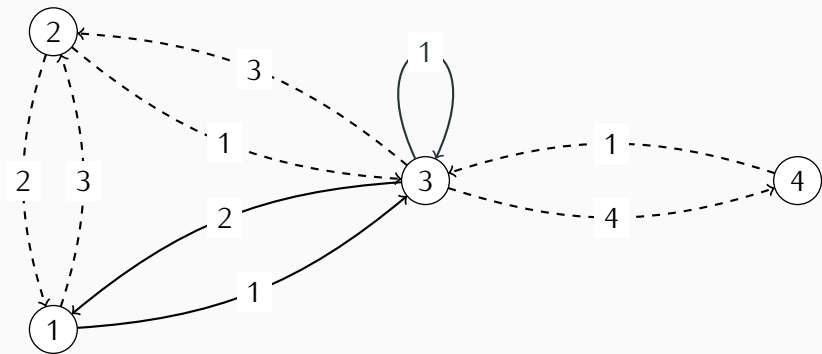
$$\theta_k \triangleq \mathbb{E}_{\mu_{v_{k+1} v_k \sigma_k}}[p_0(A_{\sigma_1} \cdots A_{\sigma_k} x)].$$

We have

$$\theta_k \geq \frac{\gamma^{2d}}{d^{-(v_k)}} \theta_{k-1}.$$

$$\frac{\gamma}{\|\mathbf{A}^\top\|_\infty^{\frac{1}{2d}}} \leq \lim_{k \rightarrow \infty} \|A_{\sigma_1} \cdots A_{\sigma_k}\|_k^{\frac{1}{k}}.$$

Let's try it

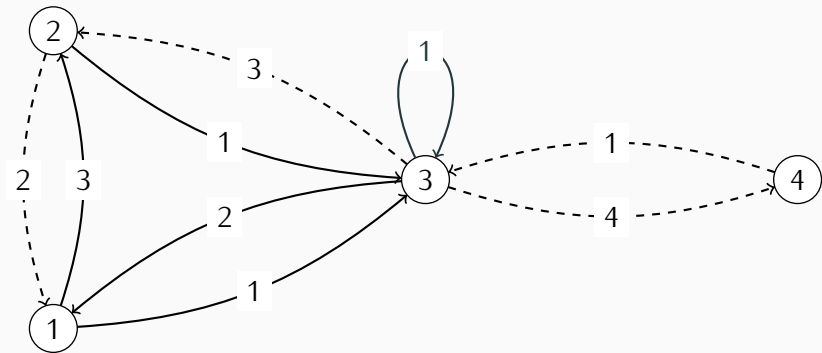


$$2d = 2$$

$3 \rightarrow_2 1 \rightarrow_1 3 \rightarrow_2 1 \rightarrow_3 1 \rightarrow_1 3 \rightarrow_1 3 \rightarrow_1 3 \rightarrow_1 3$

0.97289

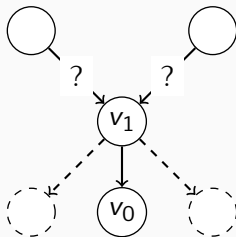
Let's try it



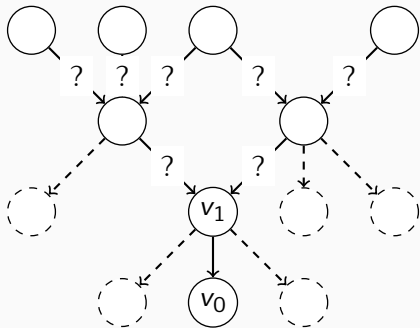
$$2d = 12$$

$3 \rightarrow_2 1 \rightarrow_1 3 \rightarrow_2 1 \rightarrow_3 2 \rightarrow_1 3 \rightarrow_1 3 \rightarrow_1 3 \rightarrow_1 3$

0.97482



Improvement



Improved guarantee

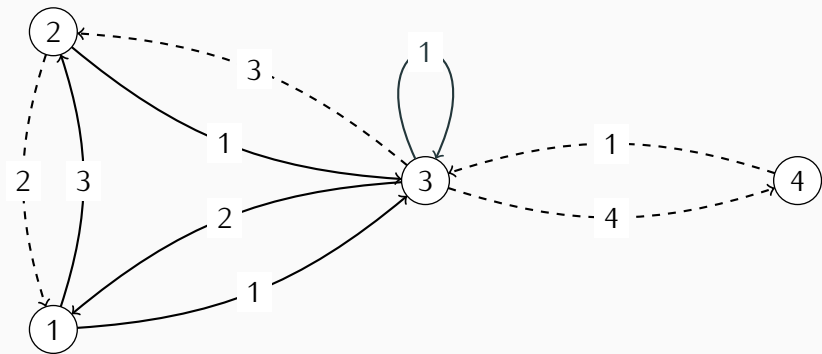
With horizon 1:

$$\frac{\underline{\gamma}}{\|\mathbf{A}^\top\|_\infty^{\frac{1}{2d}}} \leq \lim_{k \rightarrow \infty} \|A_{\sigma_1} \cdots A_{\sigma_k}\|^\frac{1}{k}.$$

With horizon l :

$$\frac{\underline{\gamma}}{\|[\mathbf{A}^\top]^l\|_\infty^{\frac{1}{2dl}}} \leq \lim_{k \rightarrow \infty} \|A_{\sigma_1} \cdots A_{\sigma_k}\|^\frac{1}{k}.$$

Let's try it



$$2d = 2, l = 3$$

$$3 \rightarrow_2 1 \rightarrow_1 3 \rightarrow_2 1 \rightarrow_3 2 \rightarrow_1 3 \rightarrow_1 3 \rightarrow_1 3 \rightarrow_1 3$$

$$0.97482$$

Guarantee for the upper bound

This yields, as $l \rightarrow \infty$,

$$\rho(\mathbf{A})^{-\frac{1}{2d}} \rho_{\text{SOS-}2d}(G, \mathcal{A}) \leq \lim_{k \rightarrow \infty} \|A_{\sigma_1} \cdots A_{\sigma_k}\|^{\frac{1}{k}} \leq \rho(G, \mathcal{A}) \leq \rho_{\text{SOS-}2d}(G, \mathcal{A}).$$

Practical improvements for lower bound:

- Use $\mathbf{A}(G')$ where edges of 0 dual variable are removed.
- Find lower bound in an high growth rate sequence produced.

B. L., R.M. Jungers and P.A. Parrilo, Generating unstable trajectories for Switched Systems via Dual Sum-Of-Squares techniques, *Proceedings of the 19th International Conference on Hybrid Systems: Computation and Control*, ACM, pp. 51–60, 2016.

Conclusion

Results

- Approximation guarantees depending on d and $\rho(A)$.
- New algorithm for producing high-growth sequences.
- Convex program provides both upper and lower bounds.

Open questions

- Improved guarantees? What is best possible?
- Is the sequence always periodic for a fixed d ? Can we bound the period ?