

Polynomial Optimization

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ERC “Back to the Roots” with Prof. Bart De Moor, STADIUS, KU Leuven

Polynomial optimization

$$\begin{array}{ll} \min_{x \in \mathbb{R}^n} & p(x) \\ \text{s.t.} & h_i(x) = 0 \quad \forall i \in \{1, \dots, m_h\} \\ & g_i(x) \leq 0 \quad \forall i \in \{1, \dots, m_g\} \end{array}$$

where p, h_i, g_i are **polynomials**.

Easy or hard ?

- If p, g_i are convex and h_i are linear, it is convex...
- ... but in general, it is **NP-hard**
- Special case with $p = 0$ and $m_g = 0$, system of polynomial equation, already **NP-hard**...

Two types of Lagrangian multipliers

Constant multipliers – local certificate – KKT system

Find $x \in \mathbb{R}^n, \lambda \in \mathbb{R}^{m_h}, \sigma \in \mathbb{R}^{m_g}$ s.t.

$$\nabla \left[p(x) + \sum_{i=1}^{m_h} \lambda_i h_i(x) + \sum_{i=1}^{m_g} \sigma_i^2 g_i(x) \right] = 0$$

Polynomial multipliers – global certificate – Putinar

Find $\gamma \in \mathbb{R}$ and polynomials $\lambda_i(x), \sigma_{i,j}(x) \in \mathbb{R}[x]$ s.t.

$$\forall x \in \mathbb{R}^n, \quad p(x) + \sum_{i=1}^{m_h} \lambda_i(x) h_i(x) + \sum_{i=1}^{m_g} \left(\sum_j \sigma_{i,j}^2(x) \right) g_i(x) = \gamma$$

Solving the KKT system

Given a system $S = \{x \mid h_i(x) = 0\}$, for any $\lambda \in \mathbb{R}[x]^m$,

$$\sum_{i=1}^m \lambda_i(x) h_i(x) = 0, \quad \forall x \in S.$$

This is the linear span of, for all $\alpha \in \mathbb{N}^n, i \in \{1, \dots, m\}$,

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} h_i$$

Macaulay matrix M_d has those of maxdegree below d as rows

Buchberger : Gaussian elimination $\rightarrow$ numerically unstable!

- Compute right null space Z_d with SVD $\rightarrow$ numerically stable!
- Mind the gap : rank of truncation of Z_d equal to rank of Z_d
- Gap condition is sufficient for finding the solution from Z_d .

Obtaining the minimizers for Sum-of-Squares

$$\forall \mathbf{x} \in \mathbb{R}^n, \quad p(\mathbf{x}) + \sum_{i=1}^{m_h} \lambda_i(\mathbf{x}) h_i(\mathbf{x}) + \sum_{i=1}^{m_g} \left(\sum_j \sigma_{i,j}^2(\mathbf{x}) \right) g_i(\mathbf{x}) = \gamma$$

We search over **dual** Lagrangian multipliers $\lambda_i(\mathbf{x})$, $\sigma_{i,j}(\mathbf{x})$. But then what's the **primal** $\mathbf{x}$? Isn't it the "dual of the dual" ?

- **Conic** dual is a symmetric **PSD** matrix of *moments* M_d .
- Positive semidefiniteness **necessary** but not **sufficient** for existence of a *measure* with these *moments*.
- **Flatness property** : Rank of **truncation** equal to rank of M_d
- Flatness **sufficient** condition for existence of an *atomic* measure with these moments.

Can you find the minimizer for the Goldstein-price function ?

MomentMatrix with row/column basis:

```
MonomialBasis([1, x[2], x[1], x[2]^2, x[1]*x[2], x[1]^2, x[2]^3, x[1]*x[2]^2,  
x[1]^2*x[2], x[1]^3, x[2]^4, x[1]*x[2]^3, x[1]^2*x[2]^2, x[1]^3*x[2], x[1]^4])
```

And entries in a 15×15 SymMatrix{Float64}:

0.999999	-0.999999	4.381387e-7	...	1.004378e-8
-0.999999	0.999999	-4.374988e-7		5.650864e-7
4.381387e-7	-4.374988e-7	2.282174e-9		9.580218e-7
0.999999	-0.999999	4.381366e-7		0.001309
-4.374973e-7	4.381360e-7	4.364830e-10		0.001473
2.280602e-9	4.350399e-10	3.053123e-9	...	0.002703
-0.999999	0.999999	-4.371625e-7		0.099747
4.381382e-7	-4.371629e-7	1.629694e-9		0.688158
4.349555e-10	1.628054e-9	3.198895e-9		0.497798
3.053181e-9	3.199220e-9	1.004332e-8		1.286501
0.999999	-0.999999	6.523214e-7	...	1766.382361
-4.371617e-7	6.523217e-7	2.263102e-7		-362.739066
1.628781e-9	2.263096e-7	4.378267e-7		2468.848686
3.198388e-9	4.378288e-7	5.650866e-7		691.956682
1.004378e-8	5.650864e-7	9.580218e-7		4051.055429

Randomized rounding with 2nd order moments

- 0th-order $\mathbb{E}[1]$ should be 1
- 1th-order $\mathbb{E}[x_i]$ should be the mean μ : previous slide uses μ as candidate
- 2th-order $\mathbb{E}[x_i x_j]$ should be the covariance Σ :

Rounded random sampling:

- Candidates sampled from $\mathcal{N}(\mu, \Sigma)$?
- Should also try to round it to the feasible set! E.g., for $x_i = \{-1, +1\}$, round with **sign**. **Guarantees** proved in [GW95].

[GW95] Goemans, M. X., Williamson, D. P. (1995). *Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming.*

How do we round for arbitrary sets ?

Example

Let P_1, P_2 be polyhedra. $x_1 \in P_1, x_2 \in P_2, a(x_2)^\top x_1 \leq \beta(x_2)$.

1. $\bar{v}_2 \leftarrow \text{project } v_2 \text{ to } P_2$
2. Project v_1 to $P_1, a(\bar{v}_2)^\top x_1 \leq \beta(\bar{v}_2)$

Algorithm

Input: Candidate v and constraints $g_i(x) \leq 0$

Partition variables x into y and z

$\bar{v}_z \leftarrow \text{recursively project } v_z \text{ onto } g_i(z) \leq 0 \text{ not depending on } y$

$\bar{v}_y \leftarrow \text{project } v_y \text{ to } g_i(y, \bar{v}_z) \leq 0 \text{ (convex thanks to partition)}$

return $\bar{v} \leftarrow \text{combine } \bar{v}_z \text{ and } \bar{v}_y$

Example of rounding

$$\begin{aligned}
 \min \quad & -x - y \\
 \text{s.t.} \quad & y \leq 2x^4 - 8x^3 + 8x^2 + 2 \\
 & y \leq 4x^4 - 32x^3 + 88x^2 - 96x + 36 \\
 & 0 \leq x \leq 3, \quad 0 \leq y \leq 4.
 \end{aligned}$$

Flatness

Round $\mu = \mathbb{E}[x_i]$

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MONIQUE LAURENT

order t	rank sequence	bound p_t^{mom}	solution extracted	bound	rounded solution
2	(1,1,4)	-7	none	-3	(-3, 0)
3	(1,2,2,4)	-6.6667	none	-3.9012	(-2.6667, 1.2345)
4	(1,1,1,1,6)	-5.5080	(2.3295, 3.1785)	-5.5080	(-2.3295, 3.1785)

TABLE 2

Moment relaxations for Example 6.23

Polynomial optimization as nonconvex QCQP

$$y^6$$

$$y^5$$

$$y^4$$

$$\max \quad x^6 y^3 + x^3 y^3 - x^3$$

$$\text{s.t.} \quad -2 \leq x \leq 3$$

$$5 \leq y \leq 7$$

$$x^3 + y^3 \leq 3$$

$$y^3$$

$$xy^3$$

$$x^2 y^3$$

$$x^3 y^3$$

$$x^4 y^3$$

$$x^5 y^3$$

$$x^6 y^3$$

$$y^2$$

$$xy^2$$

$$x^2 y^2$$

$$x^3 y^2$$

$$x^4 y^2$$

$$x^5 y^2$$

$$x^6 y^2$$

$$y$$

$$xy$$

$$x^2 y$$

$$x^3 y$$

$$x^4 y$$

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$$x^4 y^2$$

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$$x^6 y^2$$

$$y$$

$$xy$$

$$x^2 y$$

$$x^3 y$$

$$x^4 y$$

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$$xy^2$$

$$x^2 y^2$$

$$x^3 y^2$$

$$x^4 y^2$$

$$x^5 y^2$$

$$x^6 y^2$$

$$y$$

$$xy$$

$$x^2 y$$

$$x^3 y$$

$$x^4 y$$

$$x^5 y$$

$$x^6 y$$

$$1$$

$$x$$

$$x^2$$

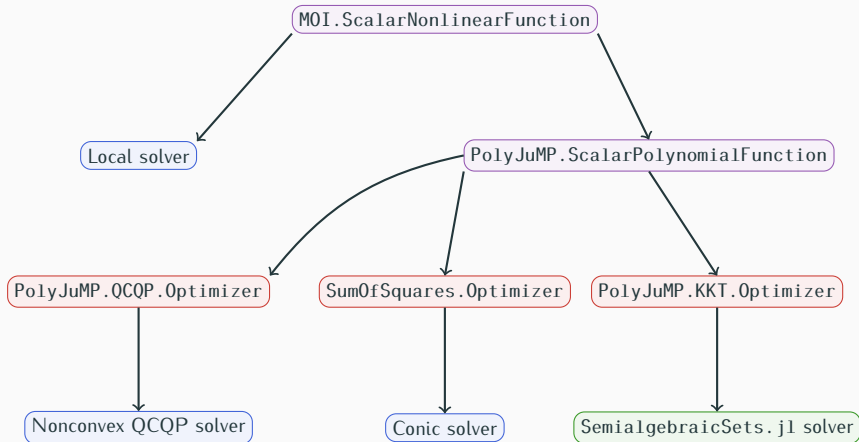
$$x^3$$

$$x^4$$

$$x^5$$

$$x^6$$

Polynomial Optimization Interface



Complementarity between Macaulay and Moment matrices

	Moment matrix	Macaulay matrix
Relies on	Conic solver	SVD
Fixed d	Polynomial	Polynomial
Growing d	Exponential	Exponential
M_d	Real radical	Spurious complex solutions
M_d	Complementary slackness	Spurious FOCs
M_d	Low-accuracy system	Numerically robust

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Seems complementary, could they work together ?

Mixing Macaulay and Sum-of-Squares frameworks

