

Polynomial Optimization

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ERC “Back to the Roots” with Prof. Bart De Moor, STADIUS, KU Leuven



Polynomial optimization

$$\begin{array}{ll} \min_{x \in \mathbb{R}^n} & p(x) \\ \text{s.t.} & h_i(x) = 0 \quad \forall i \in \{1, \dots, m_h\} \\ & g_i(x) \leq 0 \quad \forall i \in \{1, \dots, m_g\} \end{array}$$

where p, h_i, g_i are **polynomials**.

Easy or hard ?

- If p, g_i are convex and h_i are linear, it is convex...
- ... but in general, it is **NP-hard**
- Special case with $p = 0$ and $m_g = 0$, system of polynomial equation, already **NP-hard**...

Two types of Lagrangian multipliers

Constant multipliers – local certificate – KKT system

Find $x \in \mathbb{R}^n, \lambda \in \mathbb{R}^{m_h}, \sigma \in \mathbb{R}^{m_g}$ s.t.

$$p(x) + \sum_{i=1}^{m_h} \lambda_i h_i(x) + \sum_{i=1}^{m_g} \sigma_i^2 g_i(x)$$

Polynomial multipliers – global certificate – Putinar

Find $\gamma \in \mathbb{R}$ and polynomials $\lambda_i(x), \sigma_{i,j}(x) \in \mathbb{R}[x]$ s.t.

$$\forall x \in \mathbb{R}^n, \quad p(x) + \sum_{i=1}^{m_h} \lambda_i(x) h_i(x) + \sum_{i=1}^{m_g} \left(\sum_j \sigma_{i,j}^2(x) \right) g_i(x) = \gamma$$

Corner cases for KKT system

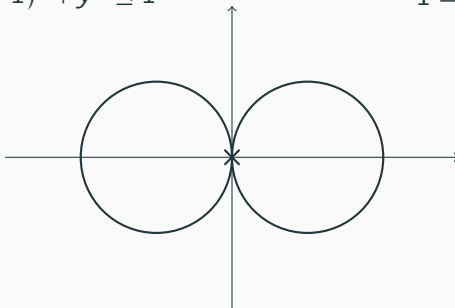
$$\min_{x \in \mathbb{R}^2} \quad x + y$$

$$\text{s.t.} \quad (x+1)^2 + y^2 \leq 1$$
$$(x-1)^2 + y^2 \leq 1$$

KKT system at $x = 0$:

$$1 + \sigma_1^2 - \sigma_2^2 = 0 \quad \partial/\partial x$$

$$1 = 0 \quad \partial/\partial y$$



What went wrong ?

Corner cases for KKT system

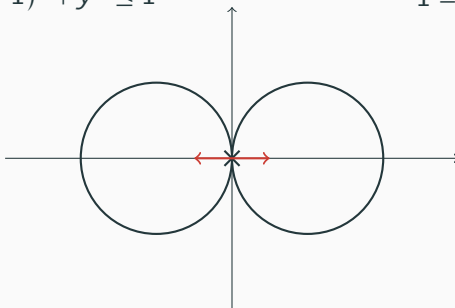
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KKT system at $x = 0$:

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What went wrong ?

The **rank** of the jacobian dropped to **1** at $x = 0$.

Solving the KKT system

Given a system $S = \{x \mid h_i(x) = 0\}$, for any $\lambda \in \mathbb{R}[x]^m$,

$$\sum_{i=1}^m \lambda_i(x) h_i(x) = 0, \quad \forall x \in S.$$

This is the linear span of, for all $\alpha \in \mathbb{N}^n, i \in \{1, \dots, m\}$,

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} h_i$$

Macaulay matrix M_d has those of maxdegree below d as rows

Buchberger : Gaussian elimination $\rightarrow$ numerically unstable!

- Compute right null space Z_d with SVD $\rightarrow$ numerically stable!
- Mind the gap : rank of truncation of Z_d equal to rank of Z_d
- Gap condition is sufficient for finding the solution from Z_d .

Obtaining the minimizers for Sum-of-Squares

$$\forall \mathbf{x} \in \mathbb{R}^n, \quad p(\mathbf{x}) + \sum_{i=1}^{m_h} \lambda_i(\mathbf{x}) h_i(\mathbf{x}) + \sum_{i=1}^{m_g} \left(\sum_j \sigma_{i,j}^2(\mathbf{x}) \right) g_i(\mathbf{x}) = \gamma$$

We search over **dual** Lagrangian multipliers $\lambda_i(\mathbf{x})$, $\sigma_{i,j}(\mathbf{x})$. But then what's the **primal** $\mathbf{x}$? Isn't it the "dual of the dual" ?

- **Conic** dual is a symmetric **PSD** matrix of *moments* M_d .
- Positive semidefiniteness **necessary** but not **sufficient** for existence of a *measure* with these *moments*.
- **Flatness property** : Rank of **truncation** equal to rank of M_d
- Flatness **sufficient** condition for existence of an *atomic* measure with these moments.

Can you find the minimizer for the Goldstein-price function ?

MomentMatrix with row/column basis:

```
MonomialBasis([1, x[2], x[1], x[2]^2, x[1]*x[2], x[1]^2, x[2]^3, x[1]*x[2]^2,  
x[1]^2*x[2], x[1]^3, x[2]^4, x[1]*x[2]^3, x[1]^2*x[2]^2, x[1]^3*x[2], x[1]^4])
```

And entries in a 15×15 SymMatrix{Float64}:

0.999999	-0.999999	4.381387e-7	...	1.004378e-8
-0.999999	0.999999	-4.374988e-7		5.650864e-7
4.381387e-7	-4.374988e-7	2.282174e-9		9.580218e-7
0.999999	-0.999999	4.381366e-7		0.001309
-4.374973e-7	4.381360e-7	4.364830e-10		0.001473
2.280602e-9	4.350399e-10	3.053123e-9	...	0.002703
-0.999999	0.999999	-4.371625e-7		0.099747
4.381382e-7	-4.371629e-7	1.629694e-9		0.688158
4.349555e-10	1.628054e-9	3.198895e-9		0.497798
3.053181e-9	3.199220e-9	1.004332e-8		1.286501
0.999999	-0.999999	6.523214e-7	...	1766.382361
-4.371617e-7	6.523217e-7	2.263102e-7		-362.739066
1.628781e-9	2.263096e-7	4.378267e-7		2468.848686
3.198388e-9	4.378288e-7	5.650866e-7		691.956682
1.004378e-8	5.650864e-7	9.580218e-7		4051.055429

JuMP interface has all we need

Rounding moment matrices

No flatness ? Try **heuristic** rounding (like MIP for fractional)

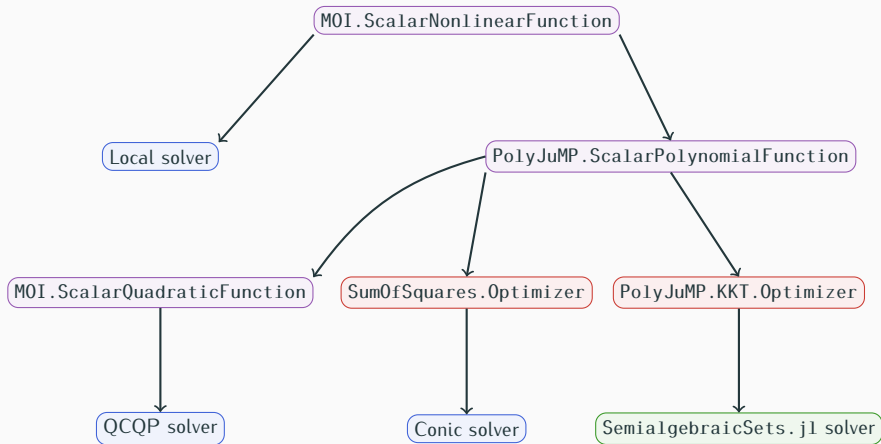
Reporting solution

- `ObjectiveBound`: Sum-of-Squares certified bound
- `ResultStatusCode`: `FEASIBLE_POINT`, `INFEASIBLE_POINT`
- Sort **multiple** solutions by status then objective value

Stateful optimizer object better than one-shot function

- **P**-time solver but **exponential** size increase with degree
- By default, `optimize!` solves one degree
- Next `optimize!` climbs one degree of hierarchy

Polynomial Optimization Interface



A bridge that can fail

Is that an issue ?

`MOI.ScalarNonlinearFunction` →

`MOI.ScalarPolynomialFunction` may fail, e.g., for $\exp(x)$.

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User chooses the solver, no default solver

These bridges are the only hope, so they are allowed to fail.



Complementarity between Macaulay and Moment matrices

	Moment matrix	Macaulay matrix
Relies on	Conic solver	SVD
Fixed d	Polynomial	Polynomial
Growing d	Exponential	Exponential
M_d	Real radical	Spurious complex solutions
M_d	Complementary slackness	Spurious FOCs
M_d	Low-accuracy system	Numerically robust

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Seems complementary, could they work together ?

Mixing Macaulay and Sum-of-Squares frameworks

